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Secret symmetry: an introduction to spontaneous symmetry breakdown and gauge fields

(1973)

1 Introduction

Here are some long-standing problems in particle theory:

(1) How can we understand the hierarchical structure of the fundamental interactions? Are the strong, medium strong (i.e. SU(3)-breaking), electromagnetic, and weak interactions truly independent, or is there some principle that establishes connections between them?

(2) How can we construct a renormalizable theory of the weak interactions, one which reproduces the low-energy successes of the Fermi theory but predicts finite higher-order corrections?

(3) How can we construct a theory of electromagnetic interactions in which electromagnetic mass differences within isotopic multiplets are finite?

(4) How can we reconcile Bjorken scaling in deep inelastic electroproduction with quantum field theory? The SLAC–MIT experiments seem to be telling us that the light-cone singularities in the product of two currents are canonical in structure; ordinary perturbation theory, on the other hand, tells us that the canonical structure is spoiled by logarithmic factors, which get worse and worse as we go to higher and higher orders in the perturbation expansion. Are there any theories of the strong interactions for which we can tame the logarithms, sum them up and show they are harmless?

Enormous progress has been made on all of these problems in the last few years. There now exists a large family of models of the weak and electromagnetic interactions that solve the second and third problem, and we have discovered a somewhat smaller family of models of the strong interactions that solve the fourth problem. As we shall see, the structure of these models is such that we are beginning to get ideas about the solution to the (very deep) first problem; connections are beginning to appear in

unexpected places, and an optimist might say that we are on the road to the first truly unified theory of the fundamental interactions. All of these marvelous developments are based upon the ideas of spontaneous symmetry breakdown and gauge fields, the subjects of these lectures.

Honesty compels me to moderate the sales-pitch of the last paragraph by pointing out that there is a fifth long-standing problem with which these theories have not yet made contact:

(5) How do we explain experiments?

We can see the reason for this embarrassing lacuna if we think a little bit more closely about the second problem, constructing a renormalizable theory of the weak interactions. At the moment, there is a plethora of such theories; they all predict that higher-order weak effects are finite, and they all predict that they are small. To find which, if any, of these theories is correct requires precision measurements of higher-order weak effects (preferably purely leptonic ones, so the strong interactions don't corrupt our predictions); these are hard to come by. Phrased another way, the Fermi theory is obviously dead wrong, because it predicts infinite higher-order corrections, but it is experimentally nearly perfect, because there are few experiments for which lowest-order Fermi theory is inadequate. Likewise for electromagnetic mass differences within isotopic multiplets: to make the differences finite, we need only to tame the high-energy behaviour of self-mass integrals; to actually compute them, though, we have to know the integrals at all energies, including the low-energy region where the strong interactions are dominant (and incalculable).

These lectures are intended as an introduction to the basic ideas of spontaneous symmetry breakdown and gauge fields, not as a survey of all the work done to date, and there are some important topics that I will not discuss at all. In particular, I will not touch at all upon the important subject of model-building; indeed, in order to simplify my examples as much as possible, I will barely mention theories involving fermions at all. Also, although I will try and make the renormalizability of the theories we discuss plausible, I will have no time to go into the guts of the renormalization problem, and therefore will say nothing about the beautiful dimensional regularization procedure of Veltman and 't Hooft, nor about the non-Abelian generalizations of the Ward identities of quantum electrodynamics, the Slavnov identities.[1]

The organization of these lectures is as follows: Section 2 is a discussion of spontaneous symmetry breakdown, Goldstone bosons, gauge fields, and the Higgs phenomenon in the simplest context, that of classical field theory. Section 3 shows how these ideas can be extended to quantum field theory in such a way that the classical reasoning of the previous section

becomes the first term in a systematic quantum expansion. The important concept of the effective potential makes its first appearance here, and its properties are discussed at length. However, an important part of the quantization program is postponed: the quantization of gauge fields. This gaping hole in the arguments of Section 3 is filled in the next two sections. Section 4 is an introduction to functional integration as a method of quantization, and Section 5 is an application of this method to gauge fields, following the ideas of Faddeev and Popov. I have tried to make Section 4 as self-contained as possible, so it may be useful to the reader who wants to learn functional integration, even if he is uninterested in the other topics of these lectures. Section 6 takes off in a new direction and explores the asymptotic properties of gauge field theories. It includes a brief review of the renormalization group.

I have learned much from conversations with Ludwig Faddeev, Howard Georgi, Sheldon Glashow, Jeffrey Goldstone, David Gross, Benjamin Lee, David Politzer, Gerard 't Hooft, Tini Veltman, Erick Weinberg, Steven Weinberg, and Frank Wilczek. Many authors who have made major contributions to this subject (including a large subset of the above) are inadequately represented in the references at the end of these lectures, because of my eccentric choice of topics and methods of approach; to these I apologize, as I do to those whom I have omitted through ignorance.[1]

2 Secret symmetries in classical field theory

2.1 *The idea of spontaneous symmetry breakdown*

In general, there is no reason why an invariance of the Hamiltonian of a quantum-mechanical system should also be an invariance of the ground state of the system. Thus, for example, the nuclear forces are rotationally invariant, but this does not mean that the ground state of a nucleus is necessarily rotationally invariant (i.e. of spin zero). This is a triviality for nuclei, but it has highly non-trivial consequences if we consider systems which, unlike nuclei, are of infinite spatial extent. The standard example is the Heisenberg ferromagnet, an infinite crystalline array of spin-$\frac{1}{2}$ magnetic dipoles, with spin–spin interactions between nearest neighbors such that neighboring dipoles tend to align. Even though the Hamiltonian is rotationally invariant, the ground state is not; it is a state in which all the dipoles are aligned in some arbitrary direction, and is infinitely degenerate for an infinite ferromagnet. A little man living inside such a ferromagnet would have a hard time detecting the rotational invariance of the laws of nature; all his experiments would be corrupted by the background magnetic field. If his experimental apparatus interacted

only weakly with the background field, he might detect rotational invariance as an approximate symmetry; if it interacted strongly, he might miss it altogether; in any case, he would have no reason to suspect that it was in fact an exact symmetry. Also, the little man would have no hope of detecting directly that the ground state in which he happens to find himself is in fact part of an infinitely degenerate multiplet. Since he is of finite extent (this is the technical meaning of 'little'), he can only change the direction of a finite number of dipoles at a time; but to go from one ground state of the ferromagnet to another, he must change the directions of an infinite number of dipoles – an impossible task.

At least at first glance, there appears to be nothing in this picture that can not be generalized to relativistic quantum mechanics. For the Hamiltonian of a ferromagnet, we can substitute the Hamiltonian of a quantum field theory; for rotational invariance, some internal symmetry; for the ground state of the ferromagnet, the vacuum state; and for the little man, ourselves. That is to say, we conjecture that the laws of nature may possess symmetries which are not manifest to us because the vacuum state is not invariant under them.[2] This situation is usually called 'spontaneous breakdown of symmetry'. The terminology is slightly deceptive, because the symmetry is not really broken, merely hidden, but we'll use it anyway.

We will begin by investigating spontaneous symmetry breakdown in the case of classical field theory. For simplicity, we will restrict ourselves to theories involving a set of n real scalar fields, which we will assemble into a real n-vector, $\boldsymbol{\phi}$, with Lagrange density[3]

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \boldsymbol{\phi}) \cdot (\partial^\mu \boldsymbol{\phi}) - U(\boldsymbol{\phi}), \tag{2.1}$$

where U is some function of the ϕs, but not of their derivatives. We will treat these theories purely classically, but use quantum-mechanical language; thus, we will call the state of lowest energy 'the vacuum', and refer to the quantities which characterize the spectra of small oscillations about the vacuum as 'particle masses'. For any of these theories, the energy density is

$$\mathscr{H} = \frac{1}{2}(\partial_0 \boldsymbol{\phi})^2 + \frac{1}{2}(\nabla \boldsymbol{\phi})^2 + U(\boldsymbol{\phi}). \tag{2.2}$$

Thus the state of lowest energy is one for which the value of $\boldsymbol{\phi}$ is a constant, which we denote by $\langle \boldsymbol{\phi} \rangle$. The value of $\langle \boldsymbol{\phi} \rangle$ is determined by the detailed dynamics of the particular theory under investigation, that is to say, by the location of the minimum (or minima) of the potential U. Sticking to our policy of using quantum language, we will call $\langle \boldsymbol{\phi} \rangle$ 'the vacuum expectation value of $\boldsymbol{\phi}$'.

Within this class of theories, it is easy to find examples for which symmetries are either manifest or spontaneously broken. The simplest one is the theory of a single field for which the potential is

$$U=\frac{\lambda}{4!}\phi^4+\frac{\mu^2}{2}\phi^2, \tag{2.3}$$

where λ is a positive number and μ^2 (despite its name) can be either positive or negative. This theory admits the symmetry

$$\phi\to-\phi. \tag{2.4}$$

If μ^2 is positive, the potential is as shown in Fig. 1. The vacuum is at $\langle\phi\rangle$ equals zero, the symmetry is manifest, and μ^2 is the mass of the scalar meson. If μ^2 is negative, though, the situation is quite different; the potential is as shown in Fig. 2. In this case, it is convenient to introduce the quantity

$$a^2=-6\mu^2/\lambda, \tag{2.5}$$

and to rewrite the potential as

$$U=\frac{\lambda}{4!}(\phi^2-a^2)^2, \tag{2.6}$$

plus an (irrelevant) constant. It is clear from this formula, and also from the figure, that the potential now has two minima, at $\phi=\pm a$. Because of the symmetry (2.4), which one we choose as the vacuum is irrelevant to the resulting physics; however, whichever one we choose, the symmetry is spontaneously broken. Let us choose $\langle\phi\rangle=a$. To investigate physics

Fig. 1

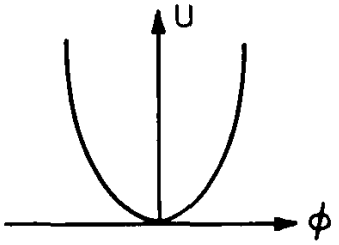
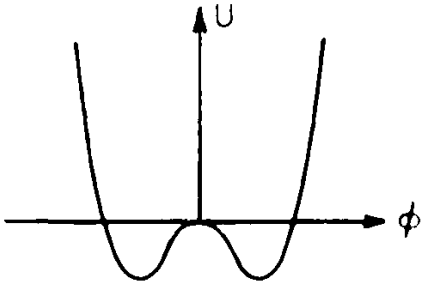


Fig. 2



about the asymmetric vacuum, let us define a new field

$$\phi' = \phi - a. \tag{2.7}$$

In terms of the new ('shifted') field,

$$\begin{aligned} U &= \frac{\lambda}{4!}(\phi'^2 + 2a\phi')^2 \\ &= \frac{\lambda}{4!}\phi'^4 + \frac{\lambda a}{6}\phi'^3 + \frac{\lambda a^2}{6}\phi'^2. \end{aligned} \tag{2.8}$$

We see that the true mass of the meson is $\lambda a^2/3$. Note that a cubic meson self-coupling has appeared as a result of the shift, which would make it hard to detect the hidden symmetry (2.4) directly.

2.2 *Goldstone bosons in an Abelian model*

A new phenomenon appears if we consider the spontaneous breakdown of continuous symmetries. Let us consider the theory of two scalar fields, A and B, with

$$U = \frac{\lambda}{4!}[A^2 + B^2 - a^2]^2. \tag{2.9}$$

This theory admits a continuous group of symmetries isomorphic to the two-dimensional rotation group, SO(2):

$$\begin{aligned} A &\to A\cos\omega + B\sin\omega, \\ B &\to -A\sin\omega + B\cos\omega. \end{aligned} \tag{2.10}$$

The minima of the potential lie on the circle

$$A^2 + B^2 = a^2. \tag{2.11}$$

Just as before, which of these we choose as the vacuum is irrelevant, but whichever one we choose, the SO(2) internal symmetry is spontaneously broken. Let us choose

$$\langle A\rangle = a,\ \langle B\rangle = 0. \tag{2.12}$$

As before, we shift the fields,

$$\phi' = \phi - \langle\phi\rangle, \tag{2.13}$$

and find

$$U = \frac{\lambda}{4!}(A'^2 + B'^2 + 2aA')^2. \tag{2.14}$$

Expanding this, we see that the A-meson has the same mass as before, but the B-meson is massless. Such a massless spinless meson is called a Goldstone boson;[4] for the class of theories under consideration, its appearance does not depend at all on the special form of the potential U,

but is a consequence only of the spontaneous breakdown of the continuous SO(2) symmetry group (2.10).

To show this, let us introduce 'angular variables',

$$A = \rho \cos \theta,$$
$$B = \rho \sin \theta. \tag{2.15}$$

In terms of these variables, (2.10) becomes

$$\rho \to \rho$$
$$\theta \to \theta + \omega, \tag{2.16}$$

and the Lagrange density becomes

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}\rho^2(\partial_\mu \theta)^2 - U(\rho). \tag{2.17}$$

In terms of these variables, SO(2) invariance is simply the statement that U does not depend on θ. The transformation to angular variables is, of course, ill-defined at the origin, and this is reflected in the singular form of the derivative part of the Lagrange density (2.17). However, this is of no interest to us, since we wish to do perturbation expansions not about the origin, but about an assumed asymmetric vacuum. With no loss of generality, we can assume this vacuum is at $\langle \rho \rangle = a$, $\langle \theta \rangle = 0$. Introducing shifted fields as before,

$$\rho' = \rho - a,$$
$$\theta' = \theta, \tag{2.18}$$

we find

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \rho')^2 + \frac{1}{2}(\rho' + a)^2(\partial_\mu \theta')^2 - U(\rho' + a). \tag{2.19}$$

It is clear from this expression that the θ-meson is massless, just because the θ-field enters the Lagrangian only through its derivatives.

This can also be seen purely geometrically, without writing down any formulae. If the vacuum is not invariant under SO(2) rotations, then there is a curve passing through the vacuum along which the potential is constant; this is the curve of points obtained from the vacuum by SO(2) rotations – in terms of our variables, the curve of constant ρ. If we expand the potential around the vacuum, no terms can appear involving the variable that measures displacement along this curve – the θ variable. Hence we always have a massless meson.

2.3 *Goldstone bosons in the general case*

This argument can easily be generalized to the spontaneous breakdown of a general continuous internal symmetry group. I will give

the generalization using somewhat more mathematical apparatus than is really necessary, in order to establish some notation that will be useful to us later on. Let us assume that we have a set of n real fields, ϕ, such that the potential is invariant under a group of transformations

$$\phi \to e^{T_a \omega^a} \phi, \tag{2.20}$$

where the Ts are a set of N real antisymmetric matrices, the group generators, the ωs are arbitrary real parameters, and the sum over repeated indices is implied. The associated infinitesimal transformations are

$$\delta\phi = T_a \delta\omega^a \phi. \tag{2.21}$$

Since the Ts are group generators, they obey the relations

$$[T_a, T_b] = c_{abc} T_c, \tag{2.22}$$

where the cs are the structure constants of the group. If we choose the Ts to be orthonormal (in the trace norm), then c^{abc} is completely antisymmetric. Invariance of the Lagrange density (2.1) implies that

$$U(\phi) = U(e^{T_a \omega^a} \phi). \tag{2.23}$$

Now let us consider the subgroup of (2.20) that leaves $\langle\phi\rangle$, the minima of U, invariant. Depending on the structure of U, this may be anything from the trivial identity subgroup (all symmetries spontaneously broken) to the full group (no symmetries spontaneously broken). In any case, though, we can always choose our group generators such that this subgroup is generated by the first M generators, where $N \geqslant M \geqslant 0$. In equations,

$$T_a \langle\phi\rangle = 0, \quad a \leqslant M. \tag{2.24}$$

By definition, the remaining $(N-M)$ generators do not leave $\langle\phi\rangle$ invariant; thus we have, passing through $\langle\phi\rangle$, an $(N-M)$-dimensional surface of constant U. Thus, by the same arguments as before, the theory must contain $(N-M)$ massless spinless mesons, one for each spontaneously broken infinitesimal symmetry. (Note that I say 'spinless', not 'scalar' or 'pseudoscalar'. The mesons may be either scalar or pseudoscalar, depending on the parity-transformation properties of the spontaneously broken generators; they may even have no well-defined parity at all, if parity is itself spontaneously broken, or if the original Lagrangian is not parity conserving.)

These mesons are called Goldstone bosons, and what we have proved in the preceding paragraph is a special case of Goldstone's theorem.[4] The theorem can be proved in much greater generality: given a field theory obeying the usual axioms (Lorentz invariance, locality, Hilbert space with positive-definite inner product, etc.), if there is a local conserved current (the axiomatic version of the statement that the Lagrangian is invariant under some continuous transformation) such that the space integral of

its time component does not annihilate the vacuum state, then the theory necessarily contains a massless spinless meson, with the same internal-symmetry and parity properties as the time component of the current.[5]

At first glance, Goldstone's theorem seems to be a killing blow to the idea that spontaneous breakdown (at least of continuous symmetries) is at work in the real world, for there is not a smidgen of experimental evidence for the existence of massless spinless mesons. However, there is one loophole: there do exist perfectly respectable field theories which do not obey the usual axioms. These are gauge field theories, of which quantum electrodynamics is the most familiar. There is no gauge in which quantum electrodynamics obeys all the axioms simultaneously; if we quantize in a covariant gauge, the theory contains states of negative norm, associated with the longitudinal photons; if we quantize in a gauge in which the theory has only states of positive norm, such as radiation gauge, the theory is not covariant. We will now investigate this loophole in more detail.

2.4 *The Higgs phenomenon in the Abelian model*

I will begin by reviewing the minimal-coupling prescription of ordinary quantum electrodynamics, and its connection with gauge invariance. Let ϕ be a set of fields (not necessarily real and spinless), with dynamics determined by a Lagrange density, $\mathscr{L}(\phi, \partial_\mu \phi)$. Let $\mathscr{L}$ be invariant under a one-parameter group of transformations,

$$\phi \to e^{iQ\omega}\phi, \tag{2.25}$$

where Q is a Hermitian matrix, called the charge matrix. (Conventionally, a set of complex basis fields of definite charge is chosen, so that Q is diagonal. However, for our purposes, it will be more convenient to choose a real set of fields, so that iQ is a real antisymmetric matrix, like the Ts in Eq. (2.20).) The associated infinitesimal transformation is

$$\delta\phi = iQ\phi\delta\omega. \tag{2.26}$$

Now let us consider transformations of the same form as Eq. (2.26), but with $\delta\omega$ space-time dependent (gauge transformations). Our theory is not invariant under these transformations, since

$$\delta(\partial_\mu\phi) = iQ(\partial_\mu\phi)\delta\omega + iQ\phi\partial_\mu(\delta\omega), \tag{2.27}$$

and the second term spoils the invariance. We can take care of this, though, by enlarging the theory and introducing a new field, A_μ, the gauge field, that transforms according to

$$\delta A_\mu = -\frac{1}{e}\partial_\mu(\delta\omega), \tag{2.28}$$

where e is a free parameter, called the electric charge. If we now define

$$\mathrm{D}_\mu\phi = \partial_\mu\phi + \mathrm{i}e\mathbf{Q}A_\mu\phi, \tag{2.29}$$

then

$$\delta\mathrm{D}_\mu\phi = \mathrm{i}Q\phi\delta\omega, \tag{2.30}$$

and

$$\mathscr{L}(\phi, \mathrm{D}_\mu\phi) \tag{2.31}$$

is gauge invariant. $\mathrm{D}_\mu\phi$ is called the gauge-covariant derivative, or sometimes just the covariant derivative. Of course, the expression (2.31) by itself can not be the total Lagrange density for a physically interesting theory; it contains no terms proportional to the derivatives of A_μ, so if we vary it with respect to A_μ we obtain, not true equations of motion, but equations of constraint. To make the gauge field a true dynamical variable, we must add a term involving derivatives; the simplest gauge-invariant choice is a term proportional to $(F_{\mu\nu})^2$, where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.32}$$

By convention, A_μ is normalized such that the final Lagrange density is

$$-\frac{1}{4}(F_{\mu\nu})^2 + \mathscr{L}(\phi, \mathrm{D}_\mu\phi). \tag{2.33}$$

This is just the usual Lagrange density of minimally-coupled electrodynamics, and it has the usual physical interpretation (charged particles, massless photons, etc.), if the dynamics of the ϕ-fields are such that the symmetry (2.25) does not suffer spontaneous breakdown. But what happens if the symmetry is spontaneously broken, as in (2.10)?

This question is most easily answered if we use the angular variables defined by Eq. (2.15). We can avoid some tedious algebra by observing that Eq. (2.29) can be rewritten as

$$\mathrm{D}_\mu\phi = \partial_\mu\phi + eA_\mu\frac{\delta\phi}{\delta\omega}. \tag{2.34}$$

In this form, it can be directly applied to the angular variables. From Eq. (2.16), it follows that

$$\mathrm{D}_\mu\rho' = \partial_\mu\rho', \tag{2.35}$$

and

$$\mathrm{D}_\mu\theta' = \partial_\mu\theta' + eA_\mu.$$

Applying this to Eq. (2.19) we obtain

$$\mathscr{L} = \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2}(\partial_\mu\rho')^2 + \frac{1}{2}(\rho' + a)^2(\partial_\mu\theta + eA_\mu)^2 - U(\rho' + a). \tag{2.36}$$

It is hard to directly read off the predictions of this expression for small oscillations about the vacuum, because of the presence of quadratic cross terms, terms proportional to $A_\mu \partial^\mu \theta'$. However, these can be eliminated by introducing the new variable

$$c_\mu = A_\mu + e^{-1}\partial_\mu \theta. \tag{2.37}$$

In terms of this,

$$\mathscr{L} = -\frac{1}{4}(\partial_\mu c_\nu - \partial_\nu c_\mu)^2 + \frac{1}{2}(\partial_\mu \rho')^2 + \frac{e^2}{2}(\rho' + a)^2 (c_\mu)^2 - U(\rho' + a). \tag{2.38}$$

Since the quadratic part of the Lagrangian is now in diagonal form, we can read off the eigenmodes for small vibrations about the ground state, or, in the quantum language we have been using, the particle spectrum. We see that there is a massive scalar meson associated with the ρ'-field, whose mass depends on the form of U. There is also a massive vector meson associated with the c-field, with mass given by

$$m_c^2 = e^2 a^2. \tag{2.39}$$

But the Goldstone boson, the θ-field, has completely disappeared! This seems a little less preposterous if we count degrees of freedom. A massive vector meson has three degrees of freedom, the three spin states of a spin-one particle, while a massless vector meson has only two, the two helicity states of the photon. What has happened is that the two degrees of freedom of the massless gauge field and the one degree of freedom of the Goldstone boson have combined together to make the three degrees of freedom of the c-field. The vector meson has eaten the Goldstone boson and grown heavy.

This magic trick was discovered by Peter Higgs, and is called the Higgs phenomenon. (Actually, the terminology is unfair, since the phenomenon was discovered independently by several other investigators, but we will use it anyway, since it is awkward to talk of the Brout–Englert–Guralnik–Hagen–Higgs–Kibble phenomenon.)[6] We can gain further insight into the Higgs phenomenon if we remember the motivation for the minimal-coupling prescription – gauge invariance.

Gauge invariance tells us that our theory is invariant under transformations of the form

$$\theta \to \theta + \omega, \tag{2.40}$$

with ω an *arbitrary* function of space and time. In particular, this means we can choose ω to be minus θ, that is to say, pick our gauge in such a way that the θ-field is identically zero. The reason the Goldstone boson disappears in the gauge-invariant theory is that it was never there in the first place; the degree of freedom that would be associated with the Goldstone

boson is a mere gauge phantom, an object that can be gauged away, like a longitudinal photon.

It is now clear how to extend the Higgs phenomenon to a general internal symmetry group, like (2.20). We merely have to add extra degrees of freedom (gauge fields) to promote the whole internal symmetry group to a gauge group. If we can do this, then we can always gauge away the degrees of freedom that would correspond to Goldstone bosons, and kill the Goldstone bosons before they are born. To carry out this scheme, though, we need first to develop the theory of gauge fields for general internal symmetry groups.

2.5 *Yang–Mills fields and the Higgs phenomenon in the general case*

How do we make a general internal symmetry group a gauge group? We will follow closely our discussion of electromagnetism. We begin with a theory that is invariant under transformations of the form (2.21),

$$\delta\phi = T_a \delta\omega^a \phi. \tag{2.21}$$

Now let us consider transformations of the same form, but with $\delta\omega^a$ space-time dependent. Our theory is not invariant under these transformations, since

$$\delta(\partial_\mu \phi) = T_a \delta\omega^a \partial_\mu \phi + T_a(\partial_\mu \delta\omega^a)\phi, \tag{2.41}$$

and the second term spoils the invariance. We will try to take care of this by introducing a set of N gauge fields, A_μ^a, one for each group generator, and defining the covariant derivatives

$$\mathrm{D}_\mu \phi = \partial_\mu \phi + g T_a A_\mu^a \phi. \tag{2.42}$$

where g, like e, is a free parameter. (For the moment, we will postpone the question of whether we can choose different gs for different gauge fields.) We wish to define the transformation properties of the gauge fields such that

$$\delta(\mathrm{D}_\mu \phi) = T_a \delta\omega^a \mathrm{D}_\mu \phi. \tag{2.43}$$

It is easy to see that this implies that

$$\delta A_\mu^a = c^{abc} \delta\omega^b A_\mu^c - \frac{1}{g} \partial_\mu \delta\omega^a, \tag{2.44}$$

where the cs are the structure constants of the group, defined in Eq. (2.22). (Both terms in this expression are easy to understand. The second term is a trivial generalization of the electromagnetic gauge transformation, Eq. (2.28). The first term is necessary to insure the invariance of the gauge-field couplings under space-time *independent* transformations; it states

that, under such transformations, the gauge fields transform like the group generators. (E.g., if the gauge group is isospin, the gauge fields must form an isovector.)) It follows from Eq. (2.43) that

$$\mathscr{L}(\phi, \mathrm{D}_\mu\phi) \tag{2.45}$$

is gauge invariant.

It is a bit harder to see what is the generalization of the free electromagnetic Lagrange density, $(F_{\mu\nu})^2$. The trick is to observe that, for electromagnetism

$$(\mathrm{D}_\mu\mathrm{D}_\nu - \mathrm{D}_\nu\mathrm{D}_\mu)\phi = \mathrm{i}QF_{\mu\nu}\phi \tag{2.46}$$

From this equation, the gauge invariance of $F_{\mu\nu}$ follows directly. In our case,

$$(\mathrm{D}_\mu\mathrm{D}_\nu - \mathrm{D}_\nu\mathrm{D}_\mu)\phi = T_a F^a_{\mu\nu}\phi, \tag{2.47}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gc^{abc}A^b_\mu A^c_\nu. \tag{2.48}$$

From Eq. (2.46), it follows directly that $F^a_{\mu\nu}$ is, not gauge-invariant, but gauge-covariant,

$$\delta F^a_{\mu\nu} = c^{abc}\delta\omega^b F^c_{\mu\nu}. \tag{2.49}$$

However, the quadratic form $(F^a_{\mu\nu})^2$ is gauge-invariant, and therefore the generalization of the electromagnetic Lagrange density (2.33) is

$$-\frac{1}{4}(F^a_{\mu\nu})^2 + \mathscr{L}(\phi, \mathrm{D}_\mu\phi). \tag{2.50}$$

The first Lagrange density of this type (for the special case of the isospin group) was constructed by Yang and Mills; for this reason non-Abelian gauge fields are frequently called Yang–Mills fields.[7]

Note that for non-Abelian gauge fields, in contrast to electromagnetism, there is a non-trivial interaction even in the absence of the ϕ-fields, because of the non-linear form of $F^a_{\mu\nu}$. There is a good physical reason for this, which is most easily seen by going to a particular example. Let us imagine that the gauge group is isospin. Just as the photon couples to every field that carries non-zero charge, so the I_z gauge meson, for example, must couple to every field that carries non-zero I_z. But among these fields are the other two members of the isotriplet of gauge fields. (It is for precisely the same reason that gravitation is inherently non-linear; the gravitational field couples to everything that carries energy density, including the gravitational field itself.)

Now let us return to the postponed question of whether we can have different coupling constants for different gauge fields. If the gauge group is simple (like SU(2) or SU(3)), the generators of the group, and therefore

the gauge fields, transform irreducibly under the action of the group; therefore they must all have the same coupling constant. However, if the gauge group is a product of simple factors (like SU(2)⊗SU(2)), then the generators of different factors never mix with each other under the action of the group, and the associated gauge fields can have different coupling constants. Thus there are as many independent coupling constants as there are simple factors in the gauge group, and Eq. (2.44), for example, should properly be written as

$$\delta A_\mu^a = c^{abc}\delta\omega^b A_\mu^c - \frac{1}{g_a}\partial_\mu \delta\omega^a. \tag{2.44'}$$

(no sum on a), where g_a can take on different values for gauge fields associated with different factor groups.

Now that we have developed the classical theory of non-Abelian gauge fields, let us apply it to spontaneous symmetry breakdown. Since the entire internal symmetry group has been promoted to a gauge group, we can always choose our gauge such that the degrees of freedom that would become Goldstone bosons disappear. From our experience with the Abelian model, we would expect the gauge fields associated with the spontaneously broken symmetries to acquire masses. It is easy to see that the only relevant part of the Lagrange density (2.1) is the derivative term

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \boldsymbol{\phi})\cdot(\partial^\mu \boldsymbol{\phi}) + \ldots. \tag{2.51}$$

In the presence of the gauge fields this becomes

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \boldsymbol{\phi} + g_a A_\mu^a T_a \boldsymbol{\phi})\cdot(\partial^\mu \boldsymbol{\phi} + g_b A_\mu^b T_b \boldsymbol{\phi}) + \cdots. \tag{2.52}$$

When we shift the fields, this generates a mass term

$$\mathscr{L} = (g_a A_\mu^a T_a \langle\boldsymbol{\phi}\rangle)\cdot(g_b A_\mu^b T_b \langle\boldsymbol{\phi}\rangle) + \cdots. \tag{2.53}$$

Note that gauge fields associated with symmetries that are not spontaneously broken, that is to say, those for which

$$T_a\langle\boldsymbol{\phi}\rangle = 0, \tag{2.54}$$

remain massless. Thus, if we wish to have a theory of this type with a realistic particle spectrum, the entire gauge group must be spontaneously broken, except for a one-parameter subgroup. We identify this subgroup with electric charge, and the corresponding gauge field with the only observed massless vector meson, the photon.

2.6 *Summary and remarks*

(1) We have discovered a large family of field theories that display spontaneous breakdown of internal symmetries. If the spontaneously

broken symmetry is discrete, this causes no problems; however, if the symmetry is continuous, symmetry breakdown is associated with the appearance of Goldstone bosons. This can be cured by coupling gauge fields to the system and promoting the internal symmetry group to a gauge group; the Goldstone bosons then disappear and the gauge mesons acquire masses. It is pleasant to remember that, at the times of their inventions, both the theory of non-Abelian gauge fields and the theory of spontaneous symmetry breakdown were thought to be theoretically amusing but physically untenable, because both predicted unobserved massless particles, the gauge mesons and the Goldstone bosons. It was only later that it was discovered that each of these diseases was the other's cure.

(2) Everything we have done so far has been for classical field theory. One of the main tasks before us is to see to what extent the apparatus of this section can be extended into the quantum domain. We shall see that, at least for weak couplings, it survives substantially unchanged; in particular, all of the equations we have derived can be reinterpreted as the first terms in a systematic quantum expansion.

(3) We have not touched at all on theories with fermions. It is trivial that if we couple fermions to the scalar-meson systems we have discussed, either directly (through Yukawa couplings) or indirectly (through gauge field couplings), then the shift in the scalar fields will induce an apparent symmetry-violating term in the fermion part of the Lagrangian. A more interesting question is whether spontaneous symmetry breakdown can occur in a theory without fundamental scalar fields. For example, perhaps bilinear forms in Fermi fields can develop symmetry-breaking vacuum expectation values all by themselves. I will have nothing to say about this possibility here, not because it is not important, but because so little is known about it.[8] (There is one exactly soluble model without fundamental scalars that displays the full Goldstone–Higgs phenomenon. This is the Schwinger model, quantum electrodynamics of massless fermions in two-dimensional space-time.)[9]

(4) It is important to realize that we can make the effects of spontaneous symmetry breakdown as large or as small as we want, by appropriately fudging the parameters in our models. Thus, in the real world, some of the spontaneously broken symmetries of nature may be observed as approximate symmetries in the usual sense, and others may be totally inaccessible to direct observation. Also, of course, there is no objection to exact or approximate symmetries of the usual kind coexisting with spontaneously broken symmetries. Presumably symmetries such as nucleon number conservation, neither broken nor coupled to a massless gauge meson, are of this sort.

(5) All of this is very pretty, but what does it buy us? What is the practical use of the idea of spontaneous symmetry breakdown, even by the generous standards of practicality current among high-energy theoreticians? The answer to this question will be given in the next section, when we leave classical physics and turn to quantum field theory.

3 Secret renormalizability

3.1 *The order of the arguments*

We are going to plunge immediately into the study of spontaneous symmetry breakdown in quantum field theory, despite the fact that we know nothing of the properties of quantum non-Abelian gauge fields, even in the absence of spontaneous symmetry breakdown. Logically, this is not a good order in which to do things, but I would like to get to the heart of the matter as soon as possible. Thus, if you have a critical disposition, you should assume in this section that I am talking about symmetry breakdown in the presence of at most some Abelian gauge fields, and you should ignore my occasional remarks about the non-Abelian case. In any case, we will quantize non-Abelian gauge fields later on.

In this section, we will first review the elements of renormalization theory, without worrying about spontaneous symmetry breakdown. Then we will develop a formalism for handling symmetry breakdown, without worrying about renormalization. Finally, we will bring the two strands of argument together.

3.2 *Renormalization reviewed*[10]

In any non-trivial quantum field theory, divergent integrals appear in the perturbation expansion for the Green's functions. Renormalization is a procedure for removing these divergences, order by order in perturbation theory, by adding extra terms, called counterterms, to the Lagrangian that defines the theory. For example, let us consider the expansion of the proper four-point-function (i.e. the off-mass-shell scattering amplitude) in the theory defined by

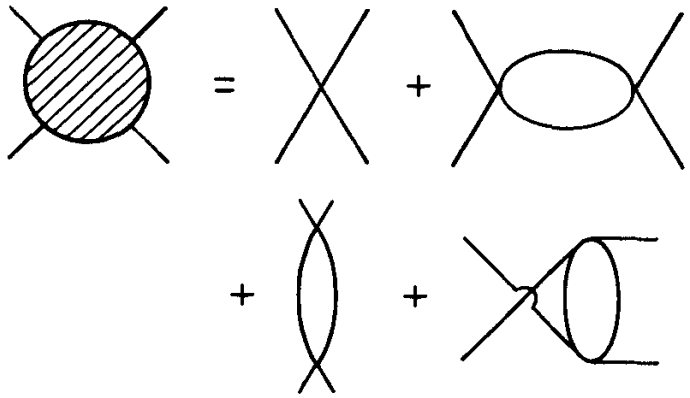
$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{3.1}$$

The first few terms in this expansion are shown in Fig. 3. All the graphs except the first correspond to divergent Feynman integrals. If we cut off the integrations at some large momentum, Λ, we obtain

$$\Gamma^{(4)} = -\lambda + a\lambda^2 \ln \Lambda + \lambda^2 f, \tag{3.2}$$

where a is a finite (i.e. cutoff-independent in the limit of large cutoff)

Fig. 3



constant and f is a finite function of the external momenta. We now change the theory, by adding an extra term (the counterterm) to $\mathscr{L}$:

$$\mathscr{L} \to \mathscr{L} - \frac{a\lambda^2}{4!} \ln \Lambda \phi^4. \tag{3.3}$$

The divergent term in Eq. (3.2) is now cancelled, and the Green's function is rendered finite. Of course, the extra term in the Lagrangian must be taken into account as an internal vertex when we compute to yet higher orders, but to this order at least, everything is OK.

It turns out that the obvious generalization of this idiotically simple manipulation gets rid of all the infinities for any field theory with polynomial interactions, to any order in perturbation theory. (I ask you to take this statement, and the ones that will follow it, on trust; they are true, but very difficult to prove.) Furthermore, it is possible to give a general rule for the counterterms that occur in each order of perturbation theory. For simplicity, I will begin by giving this rule and explaining its consequences for theories involving scalar (or pseudoscalar – parity conservation will not be assumed) and Dirac bispinor fields only. Let us write the Lagrange density of our theory in the form

$$\mathscr{L} = \mathscr{L}_0 + \sum \mathscr{L}_i, \tag{3.4}$$

where $\mathscr{L}_0$ is the standard free Lagrange density, and each $\mathscr{L}_i$ is a monomial in the fields and their derivatives. To each of these terms, let us assign a dimension, d_i, according to the rule that the dimension of a scalar field is one, of a Dirac field 3/2, and of a derivative operator, one. Thus, in Eq. (3.1), the three terms have dimensions four, two, and four respectively. (Note that we do not count dimensionful parameters, like μ^2, in computing these dimensions.) Then, to any given order of perturbation theory, all divergences can be canceled with counterterms, themselves polynomials

in the fields and their derivatives, whose dimensions obey the inequality

$$(d-4) \leqslant \sum n_i(d_i-4), \tag{3.5}$$

where n_i is the number of times $\mathscr{L}_i$ occurs in the given order.

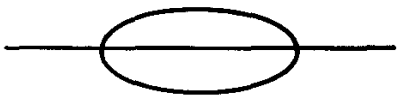
Of course, not all counterterms allowed by the inequality (3.5) are necessary. For example, the Lagrange density (3.1) is Lorentz invariant, parity invariant, and invariant under the internal symmetry $\phi \to -\phi$. Thus, unless we are so foolish as to use a cutoff procedure which breaks these symmetries, we need never worry about counterterms which are not invariant under them.

Let us check (3.5) against our sample computation. For the Lagrange density (3.1), there is only one interaction, and its dimension is four. Thus, to $O(\lambda^2)$, the order to which we worked, the right-hand side of the inequality is zero, and there are only three counterterms of appropriate dimensions and symmetry properties:

$$\mathscr{L} \to \mathscr{L} + \frac{1}{2} A(\partial_\mu \phi)^2 - \frac{1}{2} B\phi^2 - \frac{1}{4!} C\phi^4, \tag{3.6}$$

where A, B, and C are cutoff-dependent. We only saw the last of these in our sample computation, but the other two are also needed in this order, to cancel the infinities in the second order self-energy (Fig. 4).

Fig. 4



But these are not only the only counterterms to second order, they are the only ones *to general order*, because no matter how many interactions we sum up, the right-hand side of the inequality is still zero. (The new interactions induced by the counterterms themselves do not affect this argument; their dimensions are also less than or equal to four.) But these three counterterms are of the same form as the three terms in the original Lagrangian; thus they can be thought of as simply readjustments of the parameters in the original theory. (More precisely, the A term can be absorbed in a rescaling of ϕ; the B and C terms are then corrections to the mass and coupling constant.)

A theory which has this property, for which all the counterterms induced by renormalization are of the same form as terms in the original Lagrangian, is said to be renormalizable. Phrased another way, a renormalizable theory is one for which all cutoff-dependence can be removed from the

Green's functions by rescaling the fields and choosing the parameters of the theory in appropriate cutoff-dependent ways. Renormalizable theories are a very small subset of the set of all quantum field theories one can write down. (Although they may exhaust the set of theories that make sense.) For example, it is clear from our inequality (or from direct computation) that any theory involving an interaction of dimension greater than four is nonrenormalizable. However, not all theories with only interactions of dimension four or less are renormalizable. For example, the theory of mesons and nucleons interacting only through a Yukawa coupling, $\bar{\psi}\gamma_5\psi\phi$, is not renormalizable, for this interaction induces a ϕ^4 counterterm, not present in the original theory. On the other hand, the same theory with both Yukawa *and* ϕ^4 interactions *is* renormalizable. (This is a somewhat stricter definition of renormalizability than the one in common use. Most people define renormalizable to mean that there are only a finite number of counterterms induced, whether or not they were all present in the original Lagrangian.)

I have said only that the counterterms are to be chosen to cancel the infinities. This obviously leaves them undetermined, in each order, up to finite additions. For renormalizable theories, these ambiguities are usually resolved by a set of equations, called renormalization conditions, which define the scales of fields and values of renormalized masses and coupling constants in terms of Green's functions evaluated at some conventionally chosen point in momentum space. Exactly how we choose these conventions will not be relevant to our immediate purposes. For nonrenormalizable theories, in the common sense (i.e. those with an infinite number of counterterms), there are an infinite number of free parameters, which is why these theories are commonly (and properly, I think) considered disgusting.

Until now, I have said nothing about vector fields. The rules I gave for assigning dimensions to fields were in fact derived from the high-energy behavior of free propagators, as one might expect, since these are obviously the properties that control the divergences of Feynman integrals. Thus, although the dimension of a massive vector field is one, in the normal sense of dimensional analysis, its propagator is

$$-\mathrm{i}\,\frac{g_{\mu\nu}-k_\mu k_\nu/\mu^2}{k^2-\mu^2}\,. \tag{3.7}$$

Because of the second term, this grows at high momentum like the propagator for the gradient of a scalar field, an object of dimension two, and our dimension-counting formula, (3.5), breaks down. In fact, most interactions of a massive vector field are nonrenormalizable. However, if the massive

vector field is coupled to a conserved current, as if it were an Abelian gauge field, then we can shuffle variables to rewrite the theory in such a way that the propagator is

$$-\mathrm{i}\,\frac{g_{\mu\nu}-k_\mu k_\nu/k^2}{k^2-\mu^2}. \tag{3.8}$$

This grows just like a scalar propagator, so the dimension-counting procedure is good again. For a true (massless) Abelian gauge field, the theory may also be quantized in such a way that the propagator is of the form (3.8), (with μ^2 zero, of course). Thus, here also the dimension-counting procedure is good, as I trust you know from your experience with quantum electrodynamics. We shall see in Section 5 that this can also be done (with some complications) for non-Abelian gauge fields (but here only for the massless case).

However, even for quantum electrodynamics, dimension-counting is not sufficient to establish renormalizability. For example, Eq. (3.5) will certainly allow an $(A_\mu)^4$ counterterm (dimension four), but if we really had to introduce such a term into the Lagrangian, it would be a disaster – it would destroy gauge invariance. In QED, one shows such a term can not occur by a complicated sequence of arguments. (1) The theory is cut off in a cunning way that does not destroy gauge invariance. (2) Gauge invariance is used to establish relations between Green's functions, Ward identities. (3) The Ward identities are used to show that the possible gauge-noninvariant counterterms are not necessary. The same sequence of steps can be carried through for non-Abelian gauge theories, but the arguments are much more complicated; I will not have time to cover them in these lectures, and must refer you to the literature.[11]

3.3 *Functional methods and the effective potential*[12]

I would now like to put aside renormalization for the moment, and begin a new line of development, one that will lead (after an orgy of formalism) to a method for treating spontaneous symmetry breakdown in quantum field theory. For simplicity, in explaining the formalism, I will restrict myself to the theory of a single scalar field, ϕ, whose dynamics are described by a Lagrange density, $\mathscr{L}(\phi, \partial_\mu\phi)$. The generalization to more complicated cases is trivial. Let us consider the effect of adding to the Lagrange density a linear coupling of ϕ to an external source, $J(x)$, a c-number function of space and time:

$$\mathscr{L}(\phi, \partial_\mu\phi) \to \mathscr{L} + J(x)\phi(x). \tag{3.9}$$

The connected generating functional, $W(J)$, is defined in terms of the transition amplitude from the vacuum state in the far past to the vacuum

state in the far future, in the presence of the source $J(x)$,

$$e^{iW(J)} = \langle 0^+|0^-\rangle_J. \tag{3.10}$$

We can expand W in a functional Taylor series

$$W = \sum_n \frac{1}{n!} \int d^4x_1 \ldots d^4x_n G^{(n)}(x_1 \ldots x_n) J(x_1) \ldots J(x_n). \tag{3.11}$$

It is well known that the successive coefficients in this series are the connected Green's functions; $G^{(n)}$ is the sum of all connected Feynman diagrams with n external lines.

The classical field, ϕ_c, is defined by

$$\phi_c(x) = \frac{\delta W}{\delta J(x)} = \left[\frac{\langle 0^+|\phi(x)|0^-\rangle}{\langle 0^+|0^-\rangle}\right]_J. \tag{3.12}$$

The effective action, $\Gamma(\phi_c)$, is defined by a functional Legendre transformation

$$\Gamma(\phi_c) = W(J) - \int d^4x J(x)\phi_c(x). \tag{3.13}$$

From this definition, it follows directly that

$$\frac{\delta\Gamma}{\delta\phi_c(x)} = -J(x). \tag{3.14}$$

This equation will shortly turn out to be critical in the study of spontaneous breakdown of symmetry. The effective action may be expanded in a manner similar to that of (3.11):

$$\Gamma = \sum_n \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \Gamma^{(n)}(x_1 \ldots x_n)\phi_c(x_1) \ldots \phi_c(x_n). \tag{3.15}$$

It is possible to show that the successive coefficients in this series are the IPI Green's functions[13] (sometimes called proper vertices); $\Gamma^{(n)}$ is the sum of all IPI Feynman diagrams with n external lines. (An IPI (one-particle-irreducible) Feynman diagram is a connected diagram that cannot be disconnected by cutting a single internal line. By convention, IPI diagrams are evaluated with no propagators on the external lines.) There is an alternative way to expand the effective action: Instead of expanding in powers of ϕ_c, we can expand in powers of momentum (about the point where all external momenta vanish). In position space, such an expansion looks like

$$\Gamma = \int d^4x [- V(\phi_c) + \tfrac{1}{2}(\partial_\mu\phi_c)^2 Z(\phi_c) + \cdots]. \tag{3.16}$$

$V(\phi_c)$ – an ordinary function, not a functional – is called the effective potential. By comparing the expansions (3.15) and (3.16), it is easy to see that the nth derivative of V is the sum of all IPI graphs with n vanishing external momenta. In tree approximation (that is to say, neglecting all diagrams with closed loops), V is just the ordinary potential, the object we called U in Section 2.

The usual renormalization conditions of perturbation theory can be expressed in terms of the functions that occur in (3.15). For example, if we define the squared mass of the meson as the value of the inverse propagator at zero momentum, then

$$\mu^2 = \frac{d^2 V}{d\phi_c^2}\bigg|_0 . \tag{3.17a}$$

Likewise, if we define the four-point function at zero external momenta to be the coupling constant, λ, then

$$\lambda = \frac{d^4 V}{d\phi_c^4}\bigg|_0 . \tag{3.17b}$$

Similarly, the standard condition for the normalization of the field becomes

$$Z(0) = 1. \tag{3.17c}$$

We are now ready to apply this apparatus to the study of spontaneous symmetry breaking. Let us suppose our Lagrange density possesses an internal symmetry, like the classical field theories of Section 2. Then, spontaneous symmetry breaking occurs if the quantum field ϕ develops a nonzero vacuum expectation value, even when the source $J(x)$ vanishes. From Eqs. (3.12) and (3.14) this occurs if

$$\frac{\delta \Gamma}{\delta \phi_c} = 0, \tag{3.18}$$

for some non-zero value of ϕ_c. Further, since we are typically only interested in cases where the vacuum expectation value is translationally invariant (that is to say, we are not interested in the spontaneous breakdown of momentum conservation), we can simplify this to

$$\frac{dV}{d\phi_c} = 0, \tag{3.19}$$

for some non-zero value of ϕ_c. The value of ϕ_c for which the minimum occurs, which we denote by $\langle \phi \rangle$, is the expectation value of ϕ in the new (asymmetric) vacuum.

To explore the properties of the spontaneously broken theory, we define a new quantum field with vanishing vacuum expectation value,

$$\phi' = \phi - \langle \phi \rangle. \tag{3.20}$$

This generates a corresponding redefinition of the classical field,

$$\phi_c' = \phi_c - \langle\phi\rangle, \tag{3.21}$$

from which it immediately follows that the actual mass, coupling constant, etc. are computable from equations exactly like the Eqs. (3.17), except that the derivatives are evaluated at $\langle\phi\rangle$, rather than at zero. Thus, we have recreated the entire structure of our study of spontaneous symmetry breakdown in classical field theory. The only difference is that, instead of working with the classical potential U, we work with the effective potential V.

3.4 *The loop expansion*

Unfortunately, except for trivial models, we do not know the effective potential; to calculate it requires an infinite summation of Feynman diagrams, a task beyond our computational abilities. Thus, it is important to know a sensible approximation method for V. I shall now attempt to show that one such sensible method is the loop expansion: first summing all diagrams with no closed loops (tree graphs), then those with one closed loop, etc. Of course, each stage in this expansion also involves an infinite summation, but, as we shall see, this summation is trivial.

Let us introduce a parameter a into our Lagrange density, by defining

$$\mathscr{L}(\phi, \partial_\mu\phi, a) \equiv a^{-1}\mathscr{L}(\phi, \partial_\mu\phi). \tag{3.22}$$

We shall now show that the loop expansion is equivalent to a power-series expansion in a. Let P be the power of a associated with any graph. Then it is easy to see that

$$P = I - V, \tag{3.23}$$

where I is the number of internal lines in the graph and V is the number of vertices. This is because the propagator, being the inverse of the differential operator occurring in the quadratic terms in $\mathscr{L}$, carries a factor of a, while every vertex carries a factor of a^{-1}. (Note that it is important that we are dealing with IPI graphs, for which there are no propagators attached to external lines.) On the other hand, the number of loops, L, is given by

$$L = I - V + 1. \tag{3.24}$$

This is because the number of loops in a diagram is equal to the number of independent integration momenta; every internal line contributes one integration momentum, but every vertex contributes a δ function that reduces the number of independent momenta by one, except for one δ function that is left over for overall energy momentum conservation.

Combining Eqs. (3.23) and (3.24), we find that

$$P = L - 1, \tag{3.25}$$

the desired result.

The point of this analysis is not that the loop expansion is a good approximation scheme because a is a small parameter; indeed, a is equal to one. (However, it is certainly no worse than ordinary perturbation theory for small coupling constants, since the set of graphs with n loops or less certainly includes, as a subset, all graphs of nth order or less in the coupling constants.) The point is, rather, since the loop expansion corresponds to expansion in a parameter that multiplies the total Lagrange density, it is unaffected by shifts of fields, and by the redefinition of the division of the Lagrangian into free and interacting parts associated with such shifts.[14]

Thus we have a systematic expansion procedure, in any order of which we can apply the methods of Section 2. Further, the first term in the expansion of V is the classical potential, U, the negative sum of all non-derivative terms in the Lagrange density. Thus, we have not only justified in the quantum world many of the classical methods of Section 2, we have justified many of the actual computations of Section 2. They should be reliable in the corresponding quantum field theories for the usual conditions under which we expect diagrams with closed loops to be negligible, that is to say, for small coupling constants.

3.5 *A sample computation*

To put some flesh on this dry formalism, let us compute the effective potential for the theory of a single scalar field with Lagrange density

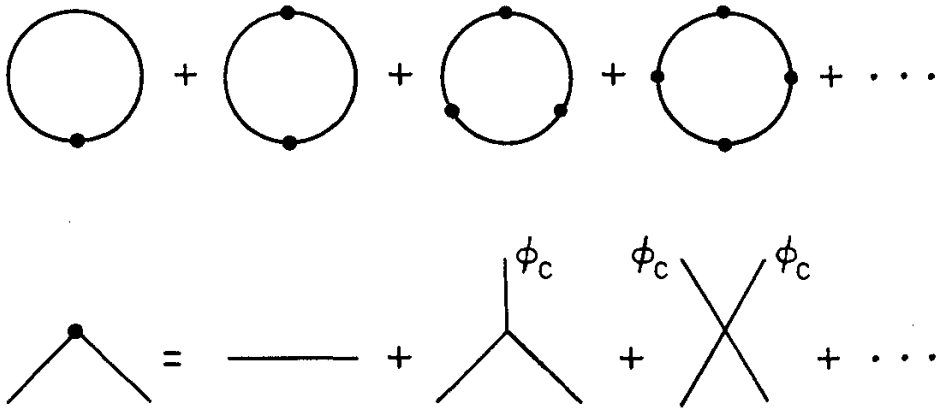
$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - U(\phi), \tag{3.26}$$

where U is a polynomial, not necessarily of renormalizable type. As stated, in the zero loop approximation,

$$V = U(\phi_c). \tag{3.27}$$

Now let us turn to the one-loop approximation. Since the one-loop approximation does not depend on how we break the Lagrangian into free and interacting parts, let us take only the first term in (3.26) as the free Lagrange density, and all of U (including possible mass terms) as the interaction. All the one-loop graphs are then shown in Fig. 5. The black dot stands for a sum of terms with zero, one, two, etc. external lines, arising from terms in U of second, third, fourth, etc. order in ϕ. (Terms linear in ϕ do not contribute to 1PI one-loop diagrams.) Each of these external lines carries zero external momentum and a factor of ϕ_c. Thus, the value of the

Fig. 5



vertex in Fig. 5 is

$$\mathrm{i}\left.\frac{\mathrm{d}^2U}{\mathrm{d}\phi^2}\right|_{\phi=\phi_c} = \mathrm{i}U''(\phi_c). \tag{3.28}$$

(The i is just the usual i from Dyson's formula.) For example, if we take the U of our old Abelian model, Eq. (2.6), then

$$U''(\phi_c) = \frac{\lambda}{6}(3\phi_c^2 - a^2). \tag{3.29}$$

Every line carries the usual massless propagator,

$$\frac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}, \tag{3.30}$$

where k is the momentum going around the loop. Thus, the sum of all the graphs in Fig. 5 corrects Eq. (3.27) in the following way:

$$V = U + \mathrm{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{1}{2n}\left(\frac{U''(\phi_c)}{k^2 + \mathrm{i}\varepsilon}\right)^n. \tag{3.31}$$

Two factors in this expression require further explanation. (1) The i in front is just a reflection of the i in the definition of W, Eq. (3.10). (2) The $1/2n$ is a combinatoric factor; rotating or reflecting the n-dot graph does not lead to a new contraction in the Wick expansion, and therefore the $1/n!$ in Dyson's formula is incompletely cancelled.

It is easy to sum this infinite series. Aside from an irrelevant (divergent) constant, the answer is

$$V = U + \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \ln(k^2 + U''(\phi_c) - \mathrm{i}\varepsilon), \tag{3.32}$$

where I have rotated the integral into Euclidean space in the standard way, but unconventionally have not dropped the $\mathrm{i}\varepsilon$. (The reason for this eccentricity will become clear shortly.) The integral is divergent; if the

integration is cut off at some large momentum Λ, we obtain

$$V = U + \frac{\Lambda^2}{32\pi^2} U'' + \frac{(U'')^2}{64\pi^2}\left(\ln\frac{U''-i\varepsilon}{\Lambda^2} - \frac{1}{2}\right), \tag{3.33}$$

plus an irrelevant constant.

The distinction between renormalizable and non-renormalizable interactions emerges very clearly in this computation. If U is a quartic polynomial (the renormalizable case), then we can remove all the cutoff-dependence from Eq. (3.33) by adding counterterms to the Lagrangian which are themselves at most quartic polynomials, and which can therefore be interpreted as corrections to the parameters in the original Lagrangian. For example, for our old Abelian model, we obtain in this way

$$V = \frac{\lambda}{4!}(\phi_c^2 - a^2)^2 + \frac{\lambda^2}{2304\pi^2}(3\phi_c^2 - a^2)^2 \ln(3\phi_c^2 - a^2 - i\varepsilon) + b\phi_c^4 + c\phi_c^2, \tag{3.34}$$

where b and c are finite constants, undetermined until we state our renormalization conventions, the conditions that define the renormalized parameters of the theory, and fix the finite parts of the counterterms.[15] (Note that it is a good thing that we retained the $i\varepsilon$, for the argument of the logarithm can become negative, and the $i\varepsilon$ is needed to tell us the sign of the imaginary part of V. We will return to this point later.)

On the other hand, if U is of quintic order or higher (the non-renormalizable case), the counterterms we must add are of yet higher order, and we are launched on the unending escalation of ambiguities that characterizes non-renormalizable theories. (A technical point: as in all renormalization schemes, the counterterms added in first order are to be considered as quantities of first order in the relevant expansion parameter. In our case, this is the (suppressed) loop-counting parameter, a, of Eq. (3.22). Thus, if we go to higher loops, the counterterms introduced at this stage are to be counted as one-loop internal parts, despite the fact that they are represented graphically by simple point vertices.)

3.6 *The most important part of this lecture*

The significant feature of the computation we have just done is that we needed to invoke no more counterterms than would have been required if there had not been spontaneous symmetry breakdown; the ultraviolet divergences of the theory respect the symmetry of the Lagrangian, even if the vacuum state does not. That this occurred in our specific computation should be no surprise; our entire formalism has been constructed so this is what happens in any computation. For $\mathscr{L}$, the Lagrange

density in Eq. (3.9), is the *total* Lagrange density for the theory. In particular, this means that it contains *all* the counterterms needed to eliminate *all* ultraviolet divergences. None of the subsequent manipulations in Section 3.3 involve any integrations over internal momenta, and therefore none of them can introduce new ultraviolet divergences.

This point is important enough to be worth stating again in a slightly different way. We have developed the theory of spontaneous symmetry breakdown in quantum field theory in such a way that we remove all the ultraviolet divergences from the theory *before* we shift the fields. Before we shift the fields, everything is manifestly symmetric under the full internal symmetry group of the theory; therefore there is no way in which asymmetric counterterms can arise.

Once more, with feeling: *the divergence structure of a renormalizable field theory is not affected by the occurrence of spontaneous symmetry breakdown.* This simple observation is the most important part of this lecture. It is the secret of the construction of renormalizable theories of the weak interactions. These theories are apparently non-renormalizable, for they involve massive vector mesons (the W-bosons) coupled to non-conserved currents. However, this is only an appearance; in actuality, the Lagrangians of these theories involve only massless gauge fields coupled minimally to conserved currents, and are perfectly renormalizable. The mass of the vector mesons and the non-conservation of the currents are a result of spontaneous symmetry breakdown.

Likewise, we see how to construct theories in which mass differences within an isotopic multiplet are finite. We begin with a theory in which the photon is part of a set of gauge mesons that couple in an isospin-symmetric way. In such a theory, one needs only isosinglet mass counterterms. Spontaneous symmetry breakdown now occurs; the friends of the photon become massive; but there is still no need for an isospin-breaking mass counterterm.

At the end of Section 2, I asked, 'What does it buy us?' We now have the wonderful answer: secret symmetry buys us secret renormalizability.

3.7 *The physical meaning of the effective potential*

In classical field theory, the ordinary potential, $U(\phi)$, is an energy density; it is the energy per unit volume for that state in which the field assumes the value ϕ. I will now show that, in quantum field theory, the effective potential, $V(\phi_c)$, is also an energy density; it is the expectation value of the energy per unit volume in a certain state for which the expectation value of the field is ϕ_c.[16] An immediate consequence of this is that, if V has several local minima, it is only the absolute minimum that corre-

sponds to the true ground state of the theory, the state of lowest energy. As a byproduct, we will obtain the essential clue to the meaning of the mysterious imaginary part of V which appeared in our sample computation.

We begin the proof by expanding $W(J)$, defined in Eq. (3.10), in the same way we expanded Γ in Eq. (3.16):

$$W = \int d^4x \{ -\mathscr{E}(J) + \tfrac{1}{2}(\partial_\mu J)^2 X(J) + \cdots \}. \tag{3.35}$$

Now let us consider a $J(x)$ which has a constant value, which we denote by J, throughout a box of side L, during a time T, and which goes to zero smoothly outside this space-time region. Under these conditions, for very large L and T, the first term in Eq. (3.35) is the dominant one, and

$$e^{iW} = \langle 0^+ | 0^- \rangle \approx e^{-iL^3 T \mathscr{E}(J)}. \tag{3.36}$$

What has happened physically is that, throughout the box, we have smoothly changed the Hamiltonian density of the theory:

$$\mathscr{H} \to \mathscr{H} - J\phi. \tag{3.37}$$

Thus we would expect the ground state of the theory, within the box, to go adiabatically into the ground state of the theory with the additional term in is Hamiltonian density. This state would evolve in time according to the Schrödinger equation; since it is a ground state, this means that it simply develops a phase. When we turn off the perturbation, the state goes adiabatically back to the ground state of the unperturbed theory, but the phase remains. Thus, $\mathscr{E}(J)$ is the energy per unit volume of the ground state of the perturbed Hamiltonian. (Of course, level crossing might take place. To be precise, we should say not 'the ground state' but 'that stationary state of the perturbed theory that is obtained from the ground state of the unperturbed theory by adiabatically turning on the perturbation'.)

I will now begin an independent line of argument, which, when combined with the above observation, will yield the desired result. For notational simplicity, I will construct this argument for ordinary quantum mechanics, not for field theory, so we will speak of energies, rather than of energy densities; the proper generalization will be obvious. Let us remember the ancient Rayleigh–Ritz variational problem: to construct a state $|a\rangle$ that is a stationary state of the quadratic form

$$\langle a | H | a \rangle, \tag{3.38}$$

under the constraint that the norm of the state be one,

$$\langle a | a \rangle = 1. \tag{3.39}$$

This problem is traditionally solved by Lagrange multipliers; one introduces a Lagrange multiplier, called E, and varies without constraint the

form

$$\langle a|(H-E)|a\rangle. \tag{3.40}$$

In this way one obtains

$$(H-E)|a\rangle=0. \tag{3.41}$$

Hence, $|a\rangle$ is an eigenstate of H with energy E.

Now let us consider a slight variation of this problem. We add an extra equation of constraint

$$\langle a|A|a\rangle = A_c, \tag{3.42}$$

where A is some Hermitian operator and A_c some number. We must now introduce two Lagrange multipliers, which I will call E and J, and vary without constraint

$$\langle a|(H-E-JA)|a\rangle. \tag{3.43}$$

We thus obtain

$$(H-E-JA)|a\rangle=0. \tag{3.44}$$

Hence $|a\rangle$ is an eigenstate of the perturbed Hamiltonian, $H-JA$, and E is its energy. Of course, this gives us E as a function of J, and we are really interested in how things depend, not on J, but on A_c. The connection between these two quantities is easily obtained by a standard formula of first-order perturbation theory,

$$A_c=\langle a|A|a\rangle=-\frac{dE}{dJ}. \tag{3.45}$$

Hence the quantity we originally set out to make stationary is given by

$$\langle a|H|a\rangle=E+JA_c=E-J\frac{dE}{dJ}. \tag{3.46}$$

It can hardly have escaped you that (with the obvious substitution of energy densities for energies and ϕ for A) this is precisely the chain of manipulations that led to the definition of the effective potential. Thus we have found the physical meaning of the effective potential;

$$V(\phi_c)=\langle a|\mathscr{H}|a\rangle, \tag{3.47}$$

for a state $|a\rangle$ such that

$$\delta\langle a|\mathscr{H}|a\rangle=0, \tag{3.48}$$

under the constraints

$$\langle a|a\rangle=1, \tag{3.49a}$$

and

$$\langle a|\phi|a\rangle=\phi_c. \tag{3.49b}$$

We can check this interpretation in another way, by reducing the four dimensions of space-time to one. The Lagrange density then becomes the

Lagrangian for a particle of unit mass, ϕ becomes x, the position of the particle, and $U(\phi)$ becomes $U(x)$, the potential in which the particle moves. Eq. (3.32) becomes

$$V = U + \frac{1}{2}\int \frac{d\omega}{2\pi} \ln(\omega^2 + U'' - i\varepsilon)$$

$$= U + \tfrac{1}{2}(U'' - i\varepsilon)^{\frac{1}{2}}. \tag{3.50}$$

This has a direct physical interpretation: classically, the particle sits in a minimum of the potential, and its energy is the value of the potential at the minimum. To get the first quantum correction to this picture, we approximate the potential near the minimum by a harmonic oscillator potential, and add the zero-point energy of the oscillator; this is the second term in Eq. (3.50).[17]

Once we know V is an energy density, we can understand the meaning of its imaginary part. When we follow an energy level as we change the parameters of a theory, it may often happen that, at a certain point, the energy level becomes unstable; at this moment the energy acquires a negative imaginary part, equal in magnitude to half the probability of decay per unit time. This can also be seen from our earlier discussion of $\mathscr{E}(J)$ in terms of the adiabatic turning-on of a perturbation. If the ground state of the unperturbed system adiabatically moves into an unstable state of the perturbed system, it will decay, and

$$\langle 0^+ | 0^- \rangle = \exp[-iL^3 T \mathscr{E}(J)], \tag{3.51}$$

will be a number with modulus less than one. Of course, for a system of infinite spatial extent, one should not speak of decay probability per unit time, any more than one speaks of energy; one speaks of decay probability per unit time per unit volume, just as one speaks of energy density. Thus the imaginary part of the effective potential is to be interpreted as half a decay probability per unit time per unit volume. (Note that the $i\varepsilon$ in Eq. (3.32) insures that the imaginary part is negative, as it must be if this interpretation is to be consistent.)[18]

3.8 *Accidental symmetry and related phenomena*

Like all perturbative expansions, the loop expansion is trustworthy only for small dimensionless coupling constants. For small coupling constants, one usually expects higher terms in a perturbation expansion to be small compared to lower terms. This is indeed the case in our sample computation; for the Abelian model, for example, the zero-loop effective potential is of order λ, and the one-loop correction is of order λ^2. Nevertheless, there are important cases in which the one-loop

corrections are more important than the tree graphs, and play the dominant role in determining the structure of spontaneous symmetry breaking.

This is because our theory may contain interactions that do not appear at all in the zero-loop approximation to the effective potential, such as Yukawa couplings or gauge-field couplings. We have not yet explicitly computed any graphs involving closed loops of virtual fermions or gauge particles, but it is obvious that their magnitude depends only on the magnitude of the Yukawa or gauge coupling constants. Since these are independent parameters of the theory, it is always possible to choose them so the one-loop graphs are more important than the zero-loop graphs, even if all coupling constants are small. Thus, for example, in the Abelian gauge model of Section 2.4, closed loops of virtual photons turn out to make a contribution to V of order e^4. (See the Appendix for the computation.) This is more important than the zero-loop effective potential if e^4 is much greater than λ, which can happen even if e and λ are both much less than one.

There are even cases in which the one-loop effective potential is important whatever the relative magnitude of the dimensionless coupling constants. This is most easily explained with a specific example. Consider an SO(3) quintuplet of scalar mesons, which we denote by ϕ^a, where a runs from 1 to 5. The transformation properties of these fields can be most simply expressed if we assemble them into a real traceless symmetric 3×3 matrix, which we denote by $\boldsymbol{\phi}$. Under an SO(3) transformation, characterized by a rotation matrix R,

$$\boldsymbol{\phi} \to R\boldsymbol{\phi}R^T. \tag{3.52}$$

In addition, we will assume invariance under the discrete symmetry

$$\boldsymbol{\phi} \to -\boldsymbol{\phi}. \tag{3.53}$$

Thus we can only have quadratic and quartic self-couplings. The only invariant quadratic form is

$$\mathrm{Tr}\,\boldsymbol{\phi}^2 = \sum (\phi^a)^2. \tag{3.54}$$

There are apparently two possible quartic couplings, $\mathrm{Tr}\,\phi^4$ and $(\mathrm{Tr}\,\phi^2)^2$; however, these are related by the tracelessness of $\boldsymbol{\phi}$:

$$\mathrm{Tr}\,\boldsymbol{\phi}^4 = \frac{1}{2}(\mathrm{Tr}\,\boldsymbol{\phi}^2)^2 = \frac{1}{2}\left[\sum (\phi^a)^2\right]^2. \tag{3.55}$$

As it is clear from the right-hand sides of these equations, both of these terms are invariant under a larger symmetry group than SO(3), to wit, SO(5). Thus, the constraints of renormalizability (no higher than fourth-order interactions) have forced the scalar meson self-interaction, and

therefore the zero-loop effective potential, to be invariant under a larger symmetry group than we started out with. This phenomenon has been dubbed accidental symmetry by Weinberg.[19] However, if the scalar mesons are coupled to a triplet of gauge fields, the gauge interaction is not forced to be (indeed, can not be) SO(5)-invariant; however, it also does not appear in the zero-loop approximation for V.

Thus, if we attempted to analyze this model in the zero-loop approximation, we would be in the soup for two reasons: (1) We would have too rich a set of vacua – an SO(5) family instead of just an SO(3) one. (2) Even if we miraculously picked the right vacuum from this over-rich set, we would find some massless scalars that were only SO(5) Goldstone bosons, and not SO(3) ones. (Weinberg calls these pseudo-Goldstone bosons.) To find the right vacuum, and to give a mass to the pseudo-Goldstone bosons, it is necessary to compute the effects of gauge-field loops.

3.9 *An alternative method of computation*

In these lectures I have stressed a method of computation in which we first compute higher-order corrections, and then shift the fields. To be honest, I must tell you that most workers in this field prefer to do things in the other order. They rewrite the Lagrangian of the theory in terms of shifted fields

$$\phi = \phi' + \langle\phi\rangle. \tag{3.56}$$

This gives them a Lagrangian with an extra free parameter for each spinless field (the value of the shift). These are fixed at some stage in the computation by demanding that the vacuum expectation values of the shifted fields vanish,

$$\langle 0|\phi'|0\rangle = 0. \tag{3.57}$$

In other words, all IPI graphs with only one external line (tadpole graphs) should sum to zero.

This is just as good a way of doing things as the way I have explained; it is equivalent to computing directly the derivative of V and demanding that it vanish, without bothering to compute V first. The only reason I have developed the theory in the way I have is a pedagogical one; in the alternative method of development, it is not so easy to see that spontaneous symmetry breakdown does not lead to asymmetric counterterms. (The only case I can think of in which our method would be clearly superior would be for a theory in which V had two local minima; in this case, we would need to know the value of V in order to determine which of them was the absolute minimum, the true vacuum.)

4 Functional integration (vulgarized)

4.1 *Integration over infinite-dimensional spaces*

Functional integration is a method for defining and manipulating integrals over function spaces, that is to say, over infinite-dimensional spaces, in the same way the ordinary integral calculus enables us to define and manipulate integrals over finite-dimensional spaces. It is useful in theoretical physics because it is possible to represent the generating functional of a quantum field theory as a functional integral. Such a representation has many virtues; from our point of view, the chief of these is that this makes it especially easy to see how the theory changes if we make non-linear transformations on its fundamental dynamical variables. The larger the set of physically interesting nonlinear transformations, the more useful is the functional-integral representation; thus it is most useful in studying non-Abelian gauge theories.

This lecture will be devoted to explaining functional integration and its connection with field theory. Our approach will be, from a mathematical viewpoint, despicable. Nothing will be proved; everything will be done by analogy, formal manipulation of ill-defined (and sometimes divergent) quantities, and handwaving. I hope that this will at least give you an idea of what is going on and teach you to manipulate functional integrals; if you want a deeper understanding, you must go elsewhere.[20]

We begin with a very simple one-dimensional integral, the Gaussian integral,

$$\int \mathrm{d}x\, \mathrm{e}^{-\frac{1}{2}ax^2} = (2\pi/a)^{\frac{1}{2}}, \tag{4.1}$$

where a is a positive real number. By analytic continuation, the formula is also true for complex a whenever the integral is defined, that is to say, whenever a has a positive real part. Eq. (4.1) can readily be generalized to n-dimensional space. We will call a vector in such a space, $\mathbf{x}$. We will denote the usual inner product of two such vectors, $\mathbf{x}$ and $\mathbf{y}$, by $(\mathbf{x}, \mathbf{y})$. Then, if A is a real symmetric positive-definite matrix,

$$\int \mathrm{d}^n\mathbf{x}\, \mathrm{e}^{-\frac{1}{2}(\mathbf{x}, A\mathbf{x})} = (2\pi)^{n/2}(\det A)^{-\frac{1}{2}}, \tag{4.2}$$

as can easily be seen by diagonalizing A. As before, this formula is also true if A is a complex symmetric matrix with positive-definite real part, by analytic continuation.

To keep from continually writing πs and ns, we define

$$(\mathrm{d}\mathbf{x}) = \mathrm{d}^n\mathbf{x}(2\pi)^{-n/2}. \tag{4.3}$$

Thus, Eq. (4.2) becomes

$$\int (d\mathbf{x}) e^{-\frac{1}{2}(\mathbf{x}, A\mathbf{x})} = (\det A)^{-\frac{1}{2}}. \tag{4.4}$$

If we can integrate Gaussians, we can integrate exponentials of general quadratic forms. Let

$$Q(\mathbf{x}) = \tfrac{1}{2}(\mathbf{x}, A\mathbf{x}) + (\mathbf{b}, \mathbf{x}) + c \tag{4.5}$$

where $\mathbf{b}$ is some vector and c is a number. Let $\bar{\mathbf{x}}$ be the minimum of Q,

$$\bar{\mathbf{x}} = -A^{-1}\mathbf{b}. \tag{4.6}$$

Then

$$Q(\mathbf{x}) = Q(\bar{\mathbf{x}}) + \frac{1}{2}(\mathbf{x} - \bar{\mathbf{x}}, A[\mathbf{x} - \bar{\mathbf{x}}]), \tag{4.7}$$

and

$$Q(\bar{\mathbf{x}}) = -\frac{1}{2}(\mathbf{b}, A^{-1}\mathbf{b}) + c. \tag{4.8}$$

Whence,

$$\int (d\mathbf{x}) e^{-Q(\mathbf{x})} = e^{-Q(\bar{\mathbf{x}})} (\det A)^{-\frac{1}{2}}. \tag{4.9}$$

Once we have Eq. (4.9), we can do the integral of any polynomial times the exponential of a quadratic form, just by differentiating with respect to $\mathbf{b}$,

$$\int (d\mathbf{x}) P(\mathbf{x}) e^{-Q(\mathbf{x})} = P\left(-\frac{\partial}{\partial \mathbf{b}}\right) \int (d\mathbf{x}) e^{-Q(\mathbf{x})}. \tag{4.10}$$

It will be convenient later to also have formulae for integrating over an n-dimensional complex vector space, not in any contour-integral sense, but merely in the sense of integrating separately over imaginary and real parts. We will denote the usual Hermitian inner product in such a space by $(\mathbf{z}^*, \mathbf{w})$, and the $2n$-dimensional real integration (with appropriate factors of π inserted) by $(d\mathbf{z}^*)(d\mathbf{z})$. Then if A is a positive-definite Hermitian matrix,

$$\int (d\mathbf{z}^*)(d\mathbf{z}) e^{-(\mathbf{z}^*, A\mathbf{z})} = (\det A)^{-1}, \tag{4.11}$$

as can easily be seen by diagonalizing A. Note the change in the power of the determinant. This is because each eigenvalue of A contributes twice to the integral, once from the integration over the real part of $\mathbf{z}$, and once from the integration over the imaginary part. The missing $\frac{1}{2}$ in the exponential is just a matter of convention; its effects are absorbed in the definition of $(d\mathbf{z}^*)(d\mathbf{z})$. From this formula equations analogous to those we derived before follow directly; I will not bother to write them out explicitly.

Now comes the great leap of faith: there is nothing in our integration formulae that refers explicitly to the dimension of the vector space; therefore we boldly extend them to infinite-dimensional vector spaces. Let me be a bit more precise about how this is done, using Eq. (4.4) as an example. Given a quadratic form, $(\mathbf{x}, A\mathbf{x})$, defined by a linear operator, A, on an infinite-dimensional real Hilbert space, we first restrict the form to some finite-dimensional subspace. On this finite-dimensional subspace, both sides of Eq. (4.4), the integral and the determinant, are well-defined. We then let the finite-dimensional subspace grow, until, in the limit, it becomes the whole space. More precisely, we consider an increasing sequence of finite-dimensional subspaces such that their union contains a dense set of vectors. This limit defines both the infinite-dimensional integral and the infinite determinant. It is a deep problem to determine for what operators A the limits exist and are independent of the sequence of subspaces, but it is not one I will worry about here. We will assume in our manipulations that expressions like (4.4) are well-defined whenever we need them.

The infinite-dimensional spaces we will be most concerned with will be spaces of functions, for example, the space of functions of a single real variable. This special case has an unnecessary, but traditional, special notation associated with it. The vectors in the space are traditionally denoted not by $\mathbf{x}$, as we have been doing, but by some symbol that makes their nature as functions manifest, e.g., by $q(t)$, where t is the real variable. The inner product is written as

$$(q, q) = \int \mathrm{d}t\,[q(t)]^2. \tag{4.12}$$

Also, a function from the vector space to the real or complex numbers is called a functional, and derivatives, like those appearing in Eq. (4.10), are called variational derivatives, and denoted by expressions like $\delta/\delta q(t)$, rather than $\partial/\partial\mathbf{x}$. In field-theoretical applications, we will consider spaces of functions of four-dimensional space-time, usually denoted by expressions like $\phi(x)$, where x is a space-time point. In this case,

$$(\phi, \phi) = \int \mathrm{d}^4x\,[\phi(x)]^2, \tag{4.13}$$

and Eq. (4.4) would be written as

$$\int (\mathrm{d}\phi)\mathrm{e}^{-\frac{1}{2}(\phi, A\phi)} = (\det A)^{-\frac{1}{2}}. \tag{4.14}$$

In the cases that will most concern us, A will be an integral or differential operator.

4.2 *Functional integrals and generating functionals*

There are a large number of cases in which the generating functionals of quantum theories can be written as functional integrals. I will begin with an especially simple case, that of a single scalar field with non-derivative self-interaction. As in Section 3.3, let us write the Lagrange density for such a theory in the presence of an external c-number source, $J(x)$,

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + \mathscr{L}'(\phi) + J(x)\phi. \tag{4.15}$$

Here $\mathscr{L}'$ is the interaction, some polynomial function of ϕ. Let us consider this as a classical Lagrange density for a c-number field, and let us construct the classical action integral

$$S(\phi, J) = \int \mathrm{d}^4x\, \mathscr{L}. \tag{4.16}$$

S is a functional of the two c-number fields, ϕ and J. In Section 3.3, we also introduced the generating functional for the quantum theory, $\exp[\mathrm{i}W(J)]$, defined as the sum of all vacuum-to-vacuum graphs in the presence of the source J. I will now demonstrate the following remarkable connection between the quantum generating functional and the classical action integral:

$$\mathrm{e}^{\mathrm{i}W(J)} = N \int (\mathrm{d}\phi)\mathrm{e}^{\mathrm{i}S(\phi,J)}, \tag{4.17}$$

where N is a normalization factor, chosen such that W vanishes when J vanishes. Eq. (4.17) is a version of Feynman's sum over histories; a quantum transition amplitude is obtained by summing over all possible classical histories of the system. As it stands, Eq. (4.17) is ill-defined, even by our sloppy standards; the integrand is an awful oscillating object, nothing like the nicely damped Gaussians of Section 4.1. This problem is remedied by stating that the generating functional on the left-hand side of Eq. (4.17) is that of *Euclidean* Green's functions, and the functional integral is to be evaluated for fields in *Euclidean* space that vanish at infinity.

This prescription requires some explanation: Feynman amplitudes are defined, to begin with, for real external three-momenta and real external energies. However, we can analytically continue them to imaginary energies, by simultaneously rotating all energies (internal as well as external) by $\pi/2$ in the complex energy plane. It is trivial to verify that no singularities of the Feynman integral are encountered in the course of this rotation. (The analytic continuation can also be proved without recourse to perturbation theory, but this is the easiest way to see that it is possible.)

Thus we arrive at Euclidean momentum space – real three-momenta and imaginary energies. For any Euclidean momentum k_μ, we define the real variable k_4 by

$$k_0 = ik_4. \tag{4.18}$$

Thus,

$$k^2 = -k_E^2, \tag{4.19}$$

where

$$k_E^2 = \mathbf{k}^2 + k_4^2, \tag{4.20}$$

the standard Euclidean square of a vector. Also,

$$d^4k = id^4k_E \tag{4.21}$$

Euclidean position-space Green's functions are defined by analytically continuing the Fourier transforms of momentum-space Green's functions. So that the Fourier exponential factor, $\exp(ik\cdot x)$, will not blow up and spoil the continuation, we must rotate x_0 through minus $\pi/2$ at the same time we rotate k_0 through plus $\pi/2$. Thus we obtain

$$x_0 = -ix_4, \tag{4.22}$$

$$d^4x = -id^4x_E, \tag{4.23}$$

etc. Thus, for example, the Feynman propagator for a free scalar field of mass μ,

$$\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{k^2 - \mu^2 + i\varepsilon}, \tag{4.24}$$

becomes, in Euclidean space,

$$\Delta_E(x) = \int \frac{d^4k_E}{(2\pi)^4} e^{-ik\cdot x} \frac{1}{k_E^2 + \mu^2}. \tag{4.25}$$

Note that there is no need to retain the $i\varepsilon$ in Euclidean space. It will be important to us shortly that this function obeys

$$(-\Box_E^2 + \mu^2)\Delta_E(x) = \delta^{(4)}(x), \tag{4.26}$$

where

$$\Box_E^2 = \nabla^2 + \partial_4^2. \tag{4.27}$$

Since the integrand in Eq. (4.25) has no pole, Δ_E is the unique solution to Eq. (4.26); this is in contrast to the situation in Minkowski space, where the corresponding equation has many solutions, and the $i\varepsilon$ is needed to resolve the ambiguity.

Let us now turn to the verification of the functional-integral formula, Eq. (4.17). I will begin with the case of a free field, $\mathscr{L}' = 0$. The Minkowski-

space generating functional is

$$e^{iW}=\exp\left[-\frac{1}{2}\int d^4x\,d^4y\,J(x)\Delta_F(x-y)J(y)\right]. \tag{4.28}$$

Hence, the Euclidean generating functional is

$$\exp\left[\frac{1}{2}\int d^4x_E\,d^4y_E\,J(x)\Delta_E(x-y)J(y)\right]. \tag{4.29}$$

This takes care of the left-hand side of Eq. (4.17). As for the right-hand side,

$$\begin{aligned} iS &= i\int d^4x\left(\frac{1}{2}[(\partial_0\phi)^2-(\nabla\phi)^2-\mu^2\phi^2]+J\phi\right) \\ &= -\int d^4x_E\left(\frac{1}{2}[(\partial_4\phi)^2+(\nabla\phi)^2+\mu^2\phi^2]-J\phi\right). \end{aligned} \tag{4.30}$$

(I emphasize that this is not an analytic continuation, just a formal substitution. We are not proving that one well-defined object is an analytic continuation of another; we are *defining* the functional integrand.) Thus the functional integral is of the form (4.9), with

$$A=-\Box_E^2+\mu^2,\ b=-J,\ c=0. \tag{4.31}$$

Hence,

$$N\int(d\phi)e^{iS}=N(\det A)^{-\frac{1}{2}}e^{\frac{1}{2}(J,A^{-1}J)}. \tag{4.32}$$

We can now determine the normalization factor, N,

$$N=(\det A)^{\frac{1}{2}}. \tag{4.33}$$

This saves us the trouble of computing the determinant. (This is a good thing, because, in cold fact, the determinant is divergent.) Thus we obtain our final answer for the integral,

$$\exp\left[\frac{1}{2}(J,A^{-1}J)\right]=\exp\left[\frac{1}{2}\int d^4x_E\,d^4y_E\,J(x)\Delta_E(x-y)J(y)\right]. \tag{4.34}$$

This is in agreement with Eq. (4.29); in this case, at least, the functional integral has reproduced the generating functional, as promised.

If we had attempted to evaluate the integral directly in Minkowski space, using (erroneously) the integral formulae of Section 4.1, we would have arrived at a similar result, except that A would have been the Klein–Gordon operator. We would then have been stymied, for we would not have known what Green's function to use for A^{-1}. The Euclidean calculation contains no such ambiguity; the right answer (Feynman's $i\varepsilon$ rule) comes about automatically as a consequence of our prescription for con-

tinuing back into Minkowski space, *after* we have done the integration. Thus, Euclidean integration is not just a mathematical nicety, but is essential if we are to obtain an unambiguous answer. From now on, I will not explicitly do the continuations into Euclidean space and out again, but simply write my integrals as if they were to be done in Minkowski space, as in Eq. (4.17). You should always remember, though, that this is just a notational convention; really we are always integrating over Euclidean fields.

Now for the interacting case. Still being slapdash, I will ignore all questions of divergences, cutoffs, and renormalizations, and simply write down Dyson's formula for the generating functional:

$$e^{iW} = N'T\left\langle 0\middle|\exp\left[i\int(\mathscr{L}'(\phi_I)+J\phi_I)d^4x\right]\middle|0\right\rangle, \tag{4.35}$$

where $|0\rangle$ is the bare vacuum, ϕ_I is the interaction-picture field, T is the time-ordering symbol, and N' is a normalization factor, chosen as before. This can be written as

$$\begin{aligned} &N'\exp\left[i\int d^4y\,\mathscr{L}'\left(-i\frac{\delta}{\delta J(y)}\right)\right]\left\langle 0\middle|\exp\left[i\int J\phi_I d^4x\right]\middle|0\right\rangle \\ &\quad = N'\exp\left[i\int d^4y\,\mathscr{L}'\left(-i\frac{\delta}{\delta J(y)}\right)\right]\exp[iW_0(J)], \end{aligned} \tag{4.36}$$

where W_0 is the generating functional for the free field. Now for the functional integral. We split the action into two parts,

$$S=\int d^4x\,\mathscr{L}'(\phi)+S_0(\phi, J), \tag{4.37}$$

where S_0 is the action for the free field (including the source term), the quantity denoted by S in Eq. (4.27). In the spirit of Eq. (4.11),

$$\begin{aligned} N\int(d\phi)e^{iS} &= N\exp\left[i\int d^4y\,\mathscr{L}'\left(-i\frac{\delta}{\delta J(y)}\right)\right]\int(d\phi)e^{iS_0} \\ &= N\exp\left[i\int d^4y\,\mathscr{L}'\left(-i\frac{\delta}{\delta J(y)}\right)\right]e^{iW_0}, \end{aligned} \tag{4.38}$$

by our preceding evaluation. Things equal to the same thing are equal to each other. Q.E.D.

This result generalizes immediately to a theory involving several scalar fields

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\phi^a)(\partial^\mu\phi^a)-U+J_a(x)\phi^a, \tag{4.39}$$

where U depends on the fields but not their derivatives, and the sum on

repeated indices is implied. In this case, Eq. (4.17) becomes

$$e^{iW} = N \int \prod_a (d\phi^a) e^{iS}. \tag{4.40}$$

Likewise, descending from four dimensions to one, we see that for the parallel system in particle mechanics,

$$L = \tfrac{1}{2}(\dot{q}^a)^2 - U(q^a) + J_a(t) q^a, \tag{4.41}$$

a similar formula applies,

$$e^{iW} = N \int \prod_a (dq^a) e^{iS}. \tag{4.42}$$

Of course, in this case, the action is just a single integral, not a quadruple one,

$$S = \int dt\, L. \tag{4.43}$$

We can also go backwards, from 'particles' to fields, by letting the index a run over an infinite range, and identifying the qs with the Fourier components of the fields at fixed time. Thus, Eq. (4.42) is in fact more general than Eq. (4.40); it involves no conditions on the Lorentz transformation properties of the dynamical variables, merely a condition on the way in which their time derivatives enter the Lagrangian. For this reason, I will in the future use 'particle' language when the discussion is general, and return to field language only for special cases.

4.3 *Feynman rules*

Let us return for a moment to the case of a single scalar field with non-derivative interactions. Eq. (4.38) gives a formal expression for the functional integral in this case, but, if the interaction is non-trivial, it is impossible to turn this into an explicit closed form. However, it is perfectly feasible to evaluate it perturbatively, by expanding in powers of the interaction. Such an expansion gives the ordinary Feynman rules.

This can be seen most easily with the aid of a functional identity. I will first state and prove this identity for finite-dimensional real vector spaces, and then, as usual, extend it to function spaces. Let $F(\mathbf{x})$ and $G(\mathbf{x})$ be any two numerical-valued functions on a vector space; then

$$F\left(-i\frac{\partial}{\partial \mathbf{x}}\right) G(\mathbf{x}) = G\left(-i\frac{\partial}{\partial \mathbf{y}}\right) F(\mathbf{y}) e^{i(\mathbf{x},\mathbf{y})}\Big|_{\mathbf{y}=0}. \tag{4.44}$$

The identity is most easily proved by Fourier analysis, that is to say, by taking F and G to be plane waves,

$$F = e^{i(\mathbf{a},\mathbf{x})},\ G = e^{i(\mathbf{b},\mathbf{x})}, \tag{4.45}$$

with **a** and **b** fixed vectors. Then

$$e^{(\mathbf{a},\partial/\partial\mathbf{x})}e^{i(\mathbf{b},\mathbf{x})}=e^{i(\mathbf{b},\mathbf{x}+\mathbf{a})}. \tag{4.46}$$

and

$$e^{(\mathbf{b},\partial/\partial\mathbf{y})}e^{i(\mathbf{x}+\mathbf{a},\mathbf{y})}=e^{i(\mathbf{x}+\mathbf{a},\mathbf{y}+\mathbf{b})}. \tag{4.47}$$

This proves Eq. (4.44).

Extending this to a function space, and applying it to Eq. (4.38), we obtain

$$e^{iW}=N\exp\left[\frac{1}{2}\int d^4y\,d^4x\,\Delta_F(x-y)\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)}\right]$$
$$\times\exp\left[i\int d^4x(\mathscr{L}'(\phi)+J\phi)\right]\Bigg|_{\phi=0}. \tag{4.48}$$

Here I have made the obvious substitutions of J for $\mathbf{x}$, ϕ for $\mathbf{y}$, and variational derivative for ordinary derivative. Eq. (4.48) is manifestly the Feynman rules for the vacuum-to-vacuum matrix element. (If it is not manifest to you, I suggest that you compute the first few terms in the expansion for a ϕ^4 interaction.) Note that diagrams occur in the expansion in which two fields from the same interaction vertex are linked by a propagator; the functional integral does not normal-order the interaction for us. If we wish to treat normal-ordered interactions, we must do the normal-ordering by hand, by inserting explicit counterterms into the interaction.

This argument can immediately be extended to the general case, to give 'Feynman rules' for perturbatively evaluating a functional integral of the form

$$\int\prod_a(dq^a)e^{iS}, \tag{4.49}$$

where

$$S=S_0+S', \tag{4.50}$$

and

$$S_0=-\tfrac{1}{2}(q^a,A_{ab}q^b). \tag{4.51}$$

Here A is a linear operator (independent of the qs) with positive-definite real part (after the rotation to imaginary time has been performed), and S' is an arbitrary polynomial functional of the qs, possibly involving source terms. Then, just as above, we can develop a diagrammatic expansion for the integral in powers of S', exactly like Feynman rules. Every power of S' is represented by a vertex, and the propagator, $D_F^{ab}(t,t')$, is the solution of

$$A_{ab}D_F^{bc}(t,t')=-i\delta_a^c\delta(t-t'). \tag{4.52}$$

Any ambiguity in solving this equation is to be resolved by rotating to imaginary time.

Note that if S' contains derivatives of the qs, these will just become derivatives of propagators in the expansion. The familiar problem of pushing time derivatives of quantum fields through a time-ordering operator, the problem that makes perturbation theory for derivative interactions such a combinatoric nightmare, has no counterpart here, for we have no time-ordering operator and no quantum fields, just an integral over c-number fields.

Thus, for any theory, if we can write the generating functional in the form (4.49), we can just read off the Feynman rules from S' in the most naive way, replacing every derivative of a field with a momentum factor, etc., without making any mistakes. Unfortunately, at the moment, the only theories for which we can write the generating functional in the form (4.49) are those without any derivatives in the interaction, so this observation is without immediate use. However, it will become very useful shortly.

4.4 *Derivative interactions*

There is a large class of theories with derivative interactions for which it is possible to write a functional-integral representation of the generating functional. These are theories where the Lagrangian is no more than quadratic in time derivatives,

$$L=\tfrac{1}{2}\dot{q}^a K_{ab}\dot{q}^b+L_a\dot{q}^a-U, \tag{4.53}$$

where K, L, and U are functions of the qs. The only restriction I will place on these functions is that K be invertible, so that the equation for the canonical momenta,

$$p_a=K_{ab}\dot{q}^b+L_a, \tag{4.54}$$

can be solved for the $\dot{q}$s and the Hamiltonian constructed,

$$H=\tfrac{1}{2}p_a(K^{-1})^{ab}p_b+\cdots, \tag{4.55}$$

where the triplet dots indicate terms of first and zeroth order in the ps.

For these theories, the appropriate generalization of our earlier result, Eq. (4.42), turns out to be

$$e^{iW}=N\int\prod_a(dq^a)[\det K]^{\frac{1}{2}}e^{iS}. \tag{4.56}$$

In this equation, K is to be interpreted as a linear operator on the function space, and the integral is to be interpreted in the same way our earlier (Gaussian) integrals were interpreted. Everything is to be restricted to a finite-dimensional subspace, the integral is to be done over that subspace,

and the limit is to be taken. I do not know of any short argument for this formula, and have to refer you to the literature for a proof.[21] However, I can try and make it plausible to you by showing that it obeys some simple consistency checks. (1) If K is independent of the qs, and L vanishes, this reduces to the previous case. The determinant can then be pulled out of the integral and absorbed by the normalization factor, reproducing Eq. (4.42). (2) If K is independent of the qs, but L does not vanish, then, by our earlier remarks, the Feynman rules are the naive ones, with the derivative in the interaction becoming a factor of momentum at the vertex. This may be a familiar result to you if you have ever gone through the derivation of the Feynman rules for ps-pv meson–nucleon theory, or the electrodynamics of charged scalar bosons. (3) If K does depend on the qs, things are not so simple. This may be familiar to you if you followed the discussion in the literature a few years ago about the Feynman rules for chiral Lagrangians.

(4) Finally, a Lagrangian of the form (4.48) becomes one of the same form if we change coordinates. To be more precise, let us trade the qs for new variables, which we denote by $\bar{q}^a$. Then

$$L = \tfrac{1}{2}\dot{q}^a K_{ab}\dot{q}^b + \cdots = \tfrac{1}{2}\dot{\bar{q}}^a \bar{K}_{ab}\dot{\bar{q}}^b + \cdots, \tag{4.57}$$

where

$$\bar{K}_{ab} = \frac{\partial q^d}{\partial \bar{q}^a} K_{cd} \frac{\partial q^d}{\partial \bar{q}^b}. \tag{4.58}$$

This takes care of the transformation of the Lagrangian, but we still have to change variables in the functional integral. As always, we will figure out how to do this by going back to the finite-dimensional case. Suppose, in a finite dimensional space, we change from coordinates $\mathbf{x}$ to coordinates $\bar{\mathbf{x}}$. Even though $\bar{\mathbf{x}}$ may be a non-linear function of $\mathbf{x}$, $\partial\mathbf{x}/\partial\bar{\mathbf{x}}$ is a linear operator (an $n \times n$ matrix, where n is the dimension of the space), and has a determinant. The change-of-variables formula is the familiar Jacobian formula,

$$(\mathrm{d}\mathbf{x}) = (\mathrm{d}\bar{\mathbf{x}}) \det(\partial\mathbf{x}/\partial\bar{\mathbf{x}}). \tag{4.59}$$

As always, we simply extend this to the infinite-dimensional case, obtaining

$$\begin{aligned}[\det K]^{\frac{1}{2}} \prod_a (\mathrm{d}q^a) &= [\det K]^{\frac{1}{2}} \prod_a (\mathrm{d}\bar{q}^a) \det(\partial q/\partial \bar{q}) \\ &= \prod_a (\mathrm{d}\bar{q}^a)[\det \bar{K}]^{\frac{1}{2}}.\end{aligned} \tag{4.60}$$

Thus, Eq. (4.56) is independent of our choice of coordinates.

Eq. (4.56) is sometimes written in 'Hamiltonian form',[22]

$$\mathrm{e}^{\mathrm{i}W} = N \int \prod_a (\mathrm{d}p^a)(\mathrm{d}q_a)\mathrm{e}^{\mathrm{i}S}, \tag{4.61}$$

where S is, as usual, the integral of the Lagrangian, but the Lagrangian is written as a function of the ps and qs, considered as independent variables,

$$L = p^a \dot{q}_a - H. \tag{4.62}$$

Formally, it is easy to see that this is equivalent to our earlier formula, by explicitly doing the integral over the ps. This is an integral of the exponential of a quadratic form, so it can be done with Eq. (4.9). We see that we get a determinant in front, just the one we need. In addition, in the exponential, the ps are replaced by their values at the point where S is stationary with respect to variations of the ps. This means that we must solve the equations

$$\dot{q}^a = \frac{\partial H}{\partial p_a} \tag{4.63}$$

But this just reverses the standard passage from the Lagrangian to the Hamiltonian, and recreates the Lagrangian in its original form, as a function of the qs and $\dot{q}$s.

The Hamiltonian form of the functional integral must be taken with a grain of salt. Unlike the Lagrangian form, the derivative terms do not become nicely damped exponentials when we rotate to imaginary time; they stay oscillating. Thus the Hamiltonian integral is much less well-defined than the Lagrangian one. Indeed, one can show that not even the most ingenious mathematician can make it well-defined; it is possible to find examples for which the value one assigns to (4.61) depends on whether one integrates first over the ps or first over the qs. (A simple one is $H = p^2 + q^2 + \lambda p^2 q^2$; the differences arise in perturbation theory in order λ^2.) However, there is nothing wrong with (4.61) as long as you remember that always attached to it is the rule: first integrate over the ps formally, then rotate to imaginary times.

4.5 *Fermi fields*

Everything we have done until now has been for Bose fields. What about Fermi fields? For Bose fields, we found that the generating functional could be represented as an integral over ordinary c-number fields, the classical limits of Bose fields. By analogy, we would expect that the generating functional for a theory involving Fermi fields could be written as an integral over the classical limits of Fermi fields, *anticommuting* c-number fields. Anticommuting c-numbers are notoriously objects that make strong men quail; fortunately, we will be able to circumvent the problem of defining functional integrals involving them.

Suppose we were able to define a functional integral over Fermi fields. What sort of integrals would we want to evaluate? In any theory we are interested in, the Fermi fields enter the Lagrangian at most quadratically. Thus, if we denote the Fermi field(s) by η and the conjugate field(s) by η^*, the part of the action involving Fermions is of the form

$$S_f = (\eta^*, A\eta). \tag{4.64}$$

Here A is typically the sum of two terms: a constant term, from the free fermion Lagrangian, and a term involving Bose fields, from the couplings to spinless mesons and/or gauge fields. For the moment, let us consider S_f to be the total action, and the Bose fields referred to above as external fields. (We can always integrate over them later; we know how to integrate over Bose fields.) If we were able to define a functional integral over Fermi fields, we would like to prove that

$$\langle 0^+|0^-\rangle = e^{iW} = N \int (d\eta^*)(d\eta)e^{iS_f}. \tag{4.65}$$

Now let us consider the identical integral with η a complex Bose field. In this case, we know how to do the functional integral, by Eq. (4.11),

$$\int (d\eta^*)(d\eta)e^{i(\eta^*, A\eta)} = [\det(iA)]^{-1}. \tag{4.66}$$

We also know how to directly evaluate W, by perturbation theory. W is the sum of all connected Feynman graphs. For an action of the form (4.64), these are just single-closed-loop graphs, like those drawn in Fig. 5 (except that here the lines should have arrows on them, because the field is complex). What happens to the perturbation expansion if we replace bosons by fermions? The only difference is that there is a factor of minus one for every closed Fermi loop. Every graph that contributes to W has one and only one closed loop; therefore, W is replaced by minus W, or, equivalently, the inverse determinant in Eq. (4.66) is replaced by the determinant.

Thus, we would get the right answer if

$$\int (d\eta^*)(d\eta)e^{i(\eta^*, A\eta)} = \det(iA), \tag{4.67}$$

up to a constant factor, which we can always absorb in the normalization constant, N. Therefore, we *define* the left-hand side of this equation to be equal to the right-hand side. This is a poor substitute for a deep theory of integration over anticommuting c-numbers, but it does give up a compact expression (the determinant) for a sum over Fermi closed loops, and it will turn out that this is all we will need for our purposes.

4.6 *Ghost fields*

We left the theory of derivative interactions in poor shape. It is true that we had an expression for the generating functional, Eq. (4.56), but it was not in the form of an integral of an exponential; there was a determinant sitting in front. Therefore, we could not use Eq. (4.56) to develop a diagrammatic perturbation expansion of the integral. We can now use our knowledge of Fermi fields to get the determinant up into the exponential. For, if we introduce a set of complex Fermi variables, η^a, and denote by $K^{\frac{1}{2}}$ the matrix square-root of K, then

$$[\det K]^{\frac{1}{2}} = \int (\mathrm{d}\eta^*)(\mathrm{d}\eta)\mathrm{e}^{\mathrm{i}(\eta^{a*}, K^{\frac{1}{2}}_{ab}\eta^b)}, \tag{4.68}$$

up to a multiplicative constant, which can always be absorbed in the normalization factor, N. The ηs are called ghost variables (in the field-theory case, ghost fields). They are not true dynamical variables of the system, simply devices for getting a determinant up into an exponential.

Thus, the Feynman rules for the theory can be read off from an 'effective Lagrangian',

$$L_{\mathrm{eff}} = L + L_{\mathrm{g}}, \tag{4.69}$$

where L_{g}, the ghost Lagrangian, is given by

$$L_{\mathrm{g}} = \eta^{*a} K^{\frac{1}{2}}_{ab} \eta^b. \tag{4.70}$$

It is instructive to work out in detail a field-theoretic example. Let us consider the theory of a free field coupled to an external source,

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + J\phi. \tag{4.71}$$

Let us make a change of variables to a new field, A, defined by

$$\phi = A + \tfrac{1}{2}gA^2. \tag{4.72}$$

where g is a constant. (This transformation is not invertible, but that shouldn't worry us; we're only going to do perturbation theory, and (4.72) is invertible near $\phi = 0$.) In terms of A, the Lagrange density is given by

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu A)^2(1+gA)^2 - \tfrac{1}{2}u^2A^2(1+\tfrac{1}{2}gA)^2 + JA(1+\tfrac{1}{2}gA). \tag{4.73}$$

Thus we apparently have a very complicated interaction, with g some sort of coupling constant. Of course, this interaction is just an illusion; the vacuum-to-vacuum matrix element must be the same as in our original theory. However, this is not the answer you will get if you just read the Feynman rules naively out of (4.73). The right Feynman rules are obtained from an effective Lagrange density

$$\mathscr{L}_{\mathrm{eff}} = \mathscr{L} + \mathscr{L}_{\mathrm{g}}, \tag{4.74}$$

where

$$\mathscr{L}_g = \eta^* \eta (1 + gA). \tag{4.75}$$

The unphysical nature of the ghost fields is doubly clear from this expression. (1) The ghost fields are spinless fields obeying Fermi statistics. (2) The ghost propagator has no momentum dependence; it is a constant, i.

I recommend that you compute a few things to low orders of perturbation theory, using this effective Lagrange density, to convince yourself that everything works out as it should. A good starting point is the one-point function (tadpole) to order g. This should vanish. Does it?

5 The Feynman rules for gauge field theories

5.1 *Troubles with gauge invariance*

The quantization of gauge field theories is notoriously tricky. We can get an idea of the problem if we look at the simplest gauge-invariant field theory, electromagnetism.

$$\mathscr{L} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \mathscr{L}'. \tag{5.1}$$

Let us try and derive the Feynman propagators for A_μ by straightforwardly applying the methods of Section 4, without worrying about whether electromagnetism is in fact in the class of theories we discussed there. The computation is simplified by splitting the field into (four-dimensional) transverse and longitudinal parts

$$\begin{aligned} A_\mu &= A_\mu^T + A_\mu^L \\ &= (P_{\mu\nu}^T + P_{\mu\nu}^L) A^\nu. \end{aligned} \tag{5.2}$$

where the Ps are the transverse and longitudinal projection operators; in Fourier space they are given by

$$P_{\mu\nu}^T = g_{\mu\nu} - k_\mu k_\nu / k^2, \; P_{\mu\nu}^L = k_\mu k_\nu / k^2. \tag{5.3}$$

(Remember, we are secretly doing all our computations in Euclidean space, so there is no ambiguity in dividing by k^2.) Then it is easy to see that

$$S = \int \mathrm{d}^4 x [\tfrac{1}{2}(\partial_\mu A_\nu^T)^2 + \mathscr{L}']. \tag{5.4}$$

We obtain the propagators for the transverse and longitudinal parts of the field by our standard formulae; thus

$$D_{\mu\nu}^L = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\left(-\frac{\mathrm{i}}{k^2}\right) + \frac{k_\mu k_\nu}{k^2}\begin{pmatrix} i \\ 0 \end{pmatrix}. \tag{5.5}$$

The second term is obviously unacceptable; something has gone wrong.

This debacle can be explained in two ways, either from Feynman's sum over histories or from conventional canonical quantization. (1) Sum-over-

histories explanation: Feynman says that to compute a transition amplitude you must sum over all possible histories of the system. This is normally what the functional integral does for us. However, in a gauge theory, summing over all gauge fields, $A_\mu(x)$, sums over each history an infinite number of times, because fields that are connected by a gauge transformation do not represent different histories but a single history. No wonder we got divergent nonsense! (2) Canonical explanation: To canonically quantize a dynamical system, you have to find a set of initial-value variables, *p*s and *q*s, which are complete, in the sense that their values at time zero determine the values of the dynamical variables at all times. It is only in this case that the imposition of canonical commutators at time zero will determine commutators at all times and define a quantum theory. In a gauge theory, this can never be done, because you can always make a gauge transformation that vanishes at time zero but does not vanish at some other time; thus you can never find a complete set of initial-value variables. To quantize a gauge theory, you must first pick a gauge, impose some condition that eliminates the freedom to make gauge transformations. Then, if you are clever and/or lucky in your choice of gauge, you may be able to canonically quantize the theory. Of course, physical quantities are gauge-invariant, and therefore should not depend on what gauge you pick for quantization, but this always has to be proved explicitly in every particular case. We worried about none of this; no wonder we get divergent nonsense!

Both of these explanations emphasize gauge invariance as the critical feature. I personally prefer the second to the first; the injunction to sum over histories seems to me to be incomplete, for it does not tell us what measure to use when the sum is continuous, and, as we saw in our study of derivative interactions, this is not a trivial question. However, in the first part of our investigation, I will accept a quantization method invented by Fadeev and Popov, which is inspired by the first viewpoint. Later on, I will justify the Faddeev–Popov method by appealing to canonical quantization. (Please do not think I am being original in this last step; I learned the canonical justification from Faddeev.)

5.2 *The Faddeev–Popov Ansatz*

As usual, I will begin by discussing finite-dimensional integrals and later extend the results to function spaces. Our model of the function space of a gauge field theory will be a space of $n+m$ real variables, which we denote collectively by z. We will also denote the first n of these variables by x, and the last m by y. The xs will be our finite-dimensional model of the gauge-independent variables (in electrodynamics, A_μ^{T}) and the ys of

the gauge-dependent variables (A_μ^L, in electrodynamics). We will also have a model of a gauge-invariant action, a function $S(z)$, which is independent of the y-variables,

$$\frac{\partial S}{\partial y}=0. \tag{5.6}$$

We wish to define a (finite-dimensional model of the) generating functional that avoids the divergence problems we would encounter if we integrated over all the zs. This is easy; we just integrate over the xs only, and define

$$e^{iW}=\int (dx)e^{iS}. \tag{5.7}$$

(We suppress the normalization factor for the moment.) This can also be written as

$$e^{iW}=\int (dz)e^{iS}\delta(y). \tag{5.8}$$

Here $\delta(y)$ is an m-dimensional δ-function, normalized such that

$$\int g(y)(dy)\delta(y)=g(0), \tag{5.9}$$

for any function g. Eq. (5.8) says that we integrate along the surface $y=0$. Of course, since nothing depends on the ys, we could just as well integrate along an arbitrary surface, defined by

$$y=f(x), \tag{5.10}$$

where f is an m-vector, a set of m functions of the xs. We then obtain

$$e^{iW}=\int (dz)e^{iS}\delta(y-f(x)). \tag{5.11}$$

We may not be given the surface in the form (5.10), but as the solution to some set of equations,

$$F(z)=0, \tag{5.12}$$

where F is again an m-vector, a set of m functions of the zs. It is easy to rewrite the integral in a form appropriate to this description of the surface,

$$e^{iW}=\int (dz)e^{iS}\det(\partial F/\partial y)\delta(F(z)). \tag{5.13}$$

Note that, because of the presence of the δ-function, we need only evaluate the determinant on the surface. I emphasize that Eq. (5.13) defines the same expression as Eq. (5.7), and is completely independent of our choice of the functions F.

I will now state the Faddeev–Popov[2,3] quantization procedure. Let S be the action integral for a theory involving m fields, $\phi^a(x)$ (not necessarily

all scalar). Let S be invariant under some group of gauge transformations, parametrized by a set of n real functions, $\omega^b(x)$. For such a theory, a 'gauge' is defined to be a set of n equations

$$F^b(x)=0. \tag{5.14}$$

where the Fs are functions of the ϕs, possibly differential or even non-local, such that, given any $\phi^a(x)$, there is one and only one gauge transformation that makes Eq. (5.14) true. For electrodynamics, an example of a gauge is radiation gauge, $\nabla\cdot\mathbf{A}=0$. Another example is Lorentz gauge, $\partial_\mu A^\mu=0$. (You may object that in this case the gauge transformation is not unique. This is true in Minkowski space, but remember that we are always secretly working in Euclidean space.) According to Fadeev and Popov, the theory is now quantized by declaring that

$$e^{iW}=N\int\prod_a(d\phi^a)e^{iS}\det\left(\frac{\partial F^b}{\partial\omega^c}\right)\prod_b\delta(F^b). \tag{5.15}$$

where $F^a=0$ is some gauge. This is the functional analogue of Eq. (5.13). The δ-function in Eq. (5.15) is a δ-function on function space, a δ-functional if you will; it obeys the equation

$$\int(d\phi)G(\phi)\delta(\phi)=G(0), \tag{5.16}$$

for any functional G. We will call Eq. (5.15) the Fadeev–Popov Ansatz.

Remarks. (1) The choice of gauge in the Faddeev–Popov Ansatz is equivalent to the choice of surface in the finite-dimensional integral we discussed earlier. Thus, whether the Ansatz is true or false, it is at least self-consistent; it is independent of the choice of gauge. (2) Thus, to verify the Ansatz, it suffices to verify it for just one gauge. If it is true in one gauge, it is true in any other. (3) The gauge-independence of the Ansatz depends on the action being gauge-independent. Thus, the action can not contain source terms coupled linearly to the gauge fields. However, it can contain source terms coupled to gauge-invariant objects, like $(F^a_{\mu\nu})^2$, for example. Phrased in another way, the Ansatz only gives us gauge-invariant expressions for gauge-invariant Green's functions. Since the standard wisdom is that in a gauge theory only gauge-invariant quantities are physical observables. this is no great restriction. Also, once we have settled down in some fixed gauge, there is no objection to computing non-gauge-invariant objects, like gauge-field propagators, as a preliminary step in the computation of gauge-invariant objects. (4) I remind you that everything we are doing is on a purely formal level; we are ignoring complications that may arise as a result of ultraviolet divergences. Therefore, everything we do should be taken as merely heuristic, to be checked later by more careful

analysis. The manipulation of functional integrals is more efficient than other formal methods of treating gauge theories, but it is no more rigorous.

5.3 *The application of the Ansatz*

We will begin with the simplest gauge theory, electrodynamics. Since gauge transformations for this theory are parametrized by only a single function, only one equation is needed to determine a gauge. We will choose a slight generalization of the Lorentz gauge,

$$F = \partial^\mu A_\mu - f(x), \tag{5.17}$$

where $f(x)$ is an arbitrary function. Under an infinitesimal gauge transformation, Eq. (2.28),

$$\delta F = -e^{-1}\Box^2 \delta\omega. \tag{5.18}$$

Thus,

$$\det(\delta F/\delta\omega) = \det(-e^{-1}\Box^2). \tag{5.19}$$

This is a constant and can be brought outside the integral and absorbed in the normalization. Thus we obtain

$$e^{iW} = N \int (dA)(d\psi) e^{iS} \delta(\partial^\mu A_\mu - f(x)) \tag{5.20}$$

where, to simplify notation, I have indicated by (dA) the integrals over all four components of the gauge fields, and by $(d\psi)$ the integrals over all other fields in the theory.

We still do not have the integral of an exponential, so it is hard to evaluate Eq. (5.20) perturbatively. This is easily rectified. Since the integral is independent of the function f, we can integrate it with any functional of f, $G(f)$, without changing the integral (except perhaps for a normalization, which can always be absorbed in N). Thus,

$$\begin{aligned} e^{iW} &= N \int (dA)(d\psi)(df) e^{iS} \delta(\partial^\mu A_\mu - f) G(f) \\ &= N \int (dA)(d\psi) e^{iS} G(\partial^\mu A_\mu) \end{aligned} \tag{5.21}$$

In particular, if we choose

$$G(f) = e^{-\frac{i}{2\alpha}\int d^4x f^2}, \tag{5.22}$$

where α is some real number, we find

$$e^{iW} = N \int (dA)(d\psi) e^{i\left[S - \frac{1}{2\alpha}\int d^4x (\partial^\mu A_\mu)^2\right]}. \tag{5.23}$$

Thus the outcome of our application of the Ansatz is to replace the

Lagrange density of the theory by an effective Lagrange density

$$\mathscr{L}_{\text{eff}} = \mathscr{L} - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2. \tag{5.24}$$

From this it is easy to compute the electromagnetic propagator, since

$$\int \mathrm{d}^4x(\partial^\mu A_\mu)^2 = \int \mathrm{d}^4x(\partial^\mu A_\nu^{\mathrm{L}})^2. \tag{5.25}$$

Hence, the preposterous Eq. (5.5) is replaced by

$$D_{\mu\nu} = \frac{-\mathrm{i}}{k^2}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \alpha\frac{k_\mu k_\nu}{k^2}\right]. \tag{5.26}$$

Any one of these propagators should give the same results as any other in the computation of gauge-invariant objects. (I hope you know enough about electrodynamics from other sources to recognize this as a true statement.) The choice $\alpha = 1$ yields what is usually called the Feynman-gauge propagator, $\alpha = 0$ the Landau-gauge propagator, etc. For any choice of α, the propagator has the same high-energy behaviour as that of a scalar field, and therefore the dimension-counting formulae of Section 3.1 are applicable in computing counterterms.

Now let us turn to non-Abelian gauge fields. For notational simplicity, we will restrict ourselves to the case where there is only one gauge-field coupling constant. We determine a gauge as in Eq. (5.17),

$$F^a = \partial^\mu A_\mu^a - f^a, \tag{5.27}$$

where the fs are arbitrary functions. Under an infinitesimal gauge transformation, (2.44),

$$\delta F^a = g^{-1}[-\Box^2\delta\omega^a + gc^{abc}\partial^\mu(\delta\omega^b A_\mu^c)]. \tag{5.28}$$

In contrast to the Abelian case, here the determinant is not a constant. However, just as in Section 4.6, we can write it as an integral over a set of ghost fields, scalar fields obeying Fermi statistics,

$$\det\left(\frac{\delta F^a}{\delta\omega^b}\right) = \int (\mathrm{d}\eta^*)(\mathrm{d}\eta)\mathrm{e}^{\mathrm{i}S_g}, \tag{5.29}$$

where

$$S_g = \int \mathrm{d}^4x\,\mathscr{L}_g = \int \mathrm{d}^4x(\partial_\mu\eta^{*a})(\partial^\mu\eta^a - gc^{abc}\eta^b A_\mu^c). \tag{5.30}$$

and we have chosen to absorb an overall factor of $\det g^{-1}$ into N. The ghost Lagrange density can be written in a compact way if we consider the ghosts as a set of fields that transform according to the adjoint representation of the group,

$$\mathscr{L}_g = \partial_\mu\eta^{*a}D^\mu\eta^a. \tag{5.31}$$

In this form it is clear that $\mathscr{L}_g$ is not gauge-invariant; of course, there is no reason why it should be, since it is derived from the (purposefully) non-gauge-invariant Eq. (5.27). In contrast to the example of Section 4.6, here the ghosts have a momentum-dependent propagator, that of a set of massless charged scalar fields, $i\delta^{ab}/k^2$. However, they still reveal their unphysical nature by being spinless particles obeying Fermi statistics.

The remainder of the development of the non-Abelian case is exactly the same as that of the Abelian case. Thus we arrive at the effective Lagrange density

$$\mathscr{L}_{\text{eff}} = \mathscr{L} + \mathscr{L}_g - \frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2. \tag{5.32}$$

where α is an arbitrary real number. Note that if $\mathscr{L}$ is the Lagrange density for a renormalizable field theory minimally coupled to gauge fields, every term in this expression is an interaction of renormalizable type (dimension less than or equal to four). As I explained at the end of Section 3.1, this observation is just the first step in establishing renormalizability, but it is as far as we will have time to go here.

5.4 *Justification of the Ansatz*

I will now justify the Faddeev–Popov Ansatz by showing that, in a particular gauge, it is equivalent to canonical quantization. The gauge is Arnowitt–Fickler[24] gauge (sometimes called axial gauge); it is defined by

$$F^a = A^a_3 = 0. \tag{5.33}$$

where the 3 indicates the third spatial component. Unlike Eq. (5.27), this is not Lorentz-covariant, so this is a terrible gauge for performing Feynman calculations; however, this is not our purpose. For simplicity, I will construct the proof for pure Yang–Mills fields, uncoupled to other fields; the generalization is straightforward.

First we must construct the Faddeev–Popov Ansatz:

$$\begin{aligned}\delta F^a &= -g^{-1}\partial_3 \delta\omega^a + c^{abc}\delta\omega^b A^c_3 \\ &= -g^{-1}\partial_3 \delta\omega^a.\end{aligned} \tag{5.34}$$

The second line follows from Eq. (5.33). Thus, the determinant is a constant, and can be absorbed in the normalization factor; in this gauge, *there are no ghosts*, even in the non-Abelian case. Thus, the Ansatz becomes

$$e^{iW} = N \int (dA) e^{iS} \prod_a \delta(A^a_3), \tag{5.35}$$

where (dA) indicates integration over all the As. More explicitly,

$$(dA) \prod \delta(A^a_3) = \prod (dA^a_1)(dA^a_2)(dA^a_0). \tag{5.36}$$

I remind you that

$$S=-\frac{1}{4}\int \mathrm{d}^4x(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gc^{abc}A^b_\mu A^c_\nu)^2, \tag{5.37}$$

plus source terms, which I shall not bother to write explicitly.

(This is off the main line of the argument, but it is a point that may have been worrying you: the ghosts are fictitious particles, but they do have real poles in their propagators. Therefore, it seems that states involving ghosts might contribute to the absorptive parts of gauge-invariant Green's functions. This would be disturbing if it happened; fortunately, the existence of a ghost-free gauge shows that it does not.)

It will be convenient to rewrite the Ansatz in so-called first-order form,

$$\mathrm{e}^{\mathrm{i}W}=\int (\mathrm{d}F)(\mathrm{d}A)\prod_a \delta(A^a_3)\mathrm{e}^{\mathrm{i}S'}, \tag{5.38}$$

where

$$S'=\int \mathrm{d}^4x[-\tfrac{1}{2}(F^a_{\mu\nu})^2+\tfrac{1}{4}F^{\mu\nu a}(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gc^{abc}A^b_\mu A^c_\nu)], \tag{5.39}$$

and $(\mathrm{d}F)$ denotes integration over all the Fs. The integral over the Fs is trivial and obviously reproduces Eq. (5.35). S' is also equivalent to S in the normal sense of Lagrangian dynamics; if we vary S' with respect to the Fs and As independently, we get the same equations of motion we obtain by varying S with respect to the As alone. That S' is a good action in both these senses is no coincidence; it is a consequence of the integration formula (4.9). If a dynamical variable appears in the action at most quadratically, and if the coefficient of the quadratic term is a constant, then integrating over the variable is the same as eliminating it from the action by using the Euler–Lagrange equations.

So much for the Ansatz; now let us turn to canonical quantization. Again, we will use S', the first-order action, and work in Arnowitt–Fickler gauge, setting A^a_3 equal to zero. Let us write Eq. (5.39) in such a way that the dependence on various tensor components is explicit:

$$\begin{aligned}\mathscr{L}'=&-\tfrac{1}{2}(F^a_{\mu\nu})^2+\tfrac{1}{2}F^{ija}(\partial_i A^a_j-\partial_j A^a_i+gc^{abc}A^b_i A^c_j)\\&+F^{0ia}(\partial_0 A^a_i-\partial_i A^a_0+gc^{abc}A^b_0 A^c_i)\\&+F^{03a}(-\partial_3 A^a_0)+F^{i3a}(-\partial_3 A^a_i).\end{aligned} \tag{5.40}$$

where i and j run over the range 1, 2. Note the drastic simplification of the last two terms, caused by the gauge condition. We now see that canonical quantization of (5.40) is like shooting fish in a barrel: A^a_0, F^{ija}, F^{03a}, F^{13a} are constrained variables; their Euler Lagrange equations involve no time derivatives and are, therefore, not true equations of motion but

equations of constraint, fixing the constrained variables on the initial-value surface in terms of the remaining variables, A_i^a and F^{0ia}.

Let us denote the action obtained by eliminating the constrained variables by S''; it is a functional only of A_i^a and F^{0ia}. Furthermore, it is in Hamiltonian form, with the As the canonical fields and the Fs the conjugate momentum densities. Thus, we can use Eq. (4.61) to write

$$e^{iW} = N \int \prod_a (dF^{01a})(dF^{02a})(dA_1^a)(dA_2^a)e^{iS''}. \tag{5.41}$$

However, because the constrained variables enter Eq. (5.40) at most quadratically, and because the coefficients of the quadratic terms are constants, we can equally well write this as

$$\begin{aligned} e^{iW} &= N \int (dF) \prod_a (dA_0^a)(dA_1^a)(dA_2^a)e^{iS'} \\ &= N \int (dF)(dA) \prod_a \delta(A_3^a)e^{iS'}. \end{aligned} \tag{5.42}$$

But this is the Faddeev–Popov Ansatz, Eq. (5.38). Q.E.D.

5.5 *Concluding remarks*

(1) The chain of arguments we have just constructed shows both the power and the limitations of functional-integral methods, Functional integration is a supplement to canonical quantization, not a replacement for it. For example, when writing down the Ansatz, I could well have multiplied the integrand by some function of $(F_{\mu\nu}^a)^2$. This would have been just as gauge-invariant, and just as plausible *a priori* as the original Ansatz. It would have been wrong, but there would be no way to tell this without appealing to canonical quantization. On the other hand, once we have justified the Ansatz by canonical quantization, we can use it to pass from one gauge to another with incomparable ease. In particular, we can use it to pass from a gauge in which canonical quantization is simple to a gauge in which the Feynman rules are simple.

(2) I have said this before, but it deserves emphasis: Everything we have done in this section is purely heuristic; we have paid no attention to the problems caused by ultraviolet divergences. Properly, everything should be redone with careful attention to cutoffs, renormalizations, etc. Such careful investigations have been done;[11] the result is that the heuristic arguments have not betrayed us: these theories are renormalizable; renormalization does not spoil gauge invariance; ghost states never contribute to the absorptive parts of gauge-invariant Green's functions; etc. There is one exception: in theories in which some of the gauge trans-

formations are chiral, the familiar Adler–Bell–Jackiw triangle anomalies can falsify our arguments. However, if the transformation properties of the Fermi fields are chosen such that there are no anomalies in the lowest-order triangle graphs with gauge currents at the vertices, then there are no anomalies anywhere, and everything is all right.[25]

(3) People are sometimes worried that the formal apparatus for treating spontaneous symmetry breakdown, explained in Section 3, is not gauge-invariant. This is true; the vacuum expectation value of a scalar field, the effective potential, indeed, even the Feynman propagators themselves, are not gauge-invariant objects. This is also irrelevant. In quantum electrodynamics, we continually do computations using non-gauge-invariant objects, like propagators, at intermediate stages. There is nothing wrong with this, as long as we are careful to express our final results in terms of gauge-invariant quantities, like masses and cross-sections. The occurrence of spontaneous symmetry breakdown does not affect this; the form of the effective potential and the location of its minimum are indeed gauge-dependent, but the values of masses and cross-sections computed with the aid of these objects are not.

(4) At the end of Section 3, I explained how many workers prefer to do computations in terms of shifted fields, defined by

$$\phi' = \phi - \langle\phi\rangle, \tag{5.43}$$

and to determine the parameters $\langle\phi\rangle$ at the end of the computation, by self-consistency. There is one awkwardness in doing things this way; the shift generates a bilinear scalar–vector coupling from the scalar Lagrange density:

$$\tfrac{1}{2}\mathrm{D}^\mu\phi\cdot\mathrm{D}_\mu\phi + \cdots = g\partial^\mu\phi'\cdot A^a_\mu T_a\langle\phi\rangle + \cdots \tag{5.44}$$

This coupling causes a scalar–vector mixed propagator to appear in the Feynman rules of the theory; this is no difficulty in principle, but is an annoyance in practice. Fortunately, it is possible to cancel this term by a clever choice of gauge.[26] For our gauge condition, we choose

$$F^a = \partial^\mu A^a_\mu - f^a(x) - \xi\phi'\cdot T_a\langle\phi\rangle. \tag{5.45}$$

where ξ is a number to be determined later. If we go through what should be by now familiar arguments, we obtain an effective Lagrange density of the form

$$\begin{aligned}\mathscr{L}_{\text{eff}} = \mathscr{L} &+ \partial_\mu\eta^{*a}\mathrm{D}^\mu\eta^a \\ &- g\xi\eta^{*a}\eta^b(T_b\langle\phi\rangle\cdot T_a\langle\phi\rangle + T_b\phi'\cdot T_a\langle\phi\rangle) \\ &- \frac{1}{2\alpha}(\partial^\mu A^a_\mu - \xi\phi'\cdot T_a\langle\phi\rangle)^2.\end{aligned} \tag{5.46}$$

Hence, if we choose

$$\xi = \alpha g, \tag{5.47}$$

we can cancel the annoying cross terms. Note that the interactions in this Lagrange density are still of renormalizable type, dimension less than or equal to four.

6 Asymptotic freedom

6.1 *Operator products and deep inelastic electroproduction*

The topic we are now going to discuss seems, at first glance, to have very little to do with the previous lectures. It is a topic in strong-interaction physics, that of reconciling the apparent scaling in the SLAC–MIT electroproduction experiments with the predictions of quantum field theory. I will begin by summarizing very briefly the standard lore on this problem.[27]

(1) The electroproduction experiments at Stanford measure the total cross sections for the process

$$\text{electron} + \text{nucleon} \rightarrow \text{electron} + \text{anything},$$

which is, of course, the same thing as

$$\text{virtual photon} + \text{nucleon} \rightarrow \text{anything}.$$

The process is therefore described by two kinematic variables: q^2, the mass of the virtual photon, a negative number, and E, the energy of the virtual photon in the lab frame. It is convenient to trade E for the dimensionless variable

$$x = -q^2/2mE, \tag{6.1}$$

where m is the nucleon mass. Elementary kinematics restricts x to be between zero and one. The nucleons in the experiment are unpolarized, while the virtual photons can be either transverse or longitudinal; thus the cross-section can be described in terms of two dimensionless invariants, $F_i(q^2, x)$, where i is 1 or 2. The Fs are called structure functions; the details of their definitions will not be relevant to our immediate purposes.

As $-q^2$ increases, the Fs quite rapidly lose their dependence on q^2; by $q^2 = -(2\ \text{GeV})^2$, the Fs appear to be functions of x alone, within experimental error. This phenomenon is called Bjorken scaling. There are two schools of thought on Bjorken scaling. (1) Bjorken scaling is a true asymptotic phenomenon. It will persist even if the range of q^2 is increased greatly. (2) SLAC energies are too small for us to believe that we are really in the asymptotic region. Bjorken scaling is some sort of low-energy epiphenomenon, and has nothing to do with true high-energy limits.

I will adopt the first position for this lecture, but you should be aware that this is just a matter of prejudice. The second position may well be correct; only future experiment can decide the question.

It will turn out to be convenient for our purposes to phrase matters in terms of the moments of the structure functions,

$$F_i^n(q^2) = \int_0^1 \mathrm{d}x\, x^n F_i(q^2, x). \tag{6.2}$$

The problem is: why do these moments become constants (within experimental error) as q^2 becomes large and spacelike?

(2) The operator product expansion was invented by Wilson and proved to all orders of renormalized perturbation theory by Callan and Zimmerman. It is an asymptotic expansion for the product of two local operators as the distance between them becomes small, but for our purposes it will be most convenient to express the expansion in momentum space. Let A and B be any two local operators (renormalized polynomials in canonical fields and their derivatives) and let $|a\rangle$ and $|b\rangle$ be any two states. Then,

$$\int \mathrm{e}^{\mathrm{i}q\cdot x}\,\mathrm{d}^4x\,\langle a|A(x)B(-x)|b\rangle$$

$$= \sum_C f_{ABC}(q)\langle a|C(0)|b\rangle. \tag{6.3}$$

as q goes to Euclidean infinity. The sum is over a complete set of local operators (all renormalized monomials in canonical fields and their derivatives). The expansion is useful because the rate of growth of the coefficient functions, the fs, is that given by naive dimensional analysis, modulo polynomials in $\ln q^2$. (This is true to any finite order in perturbation theory; we will later investigate whether these polynomials can pile up and change the asymptotic behavior if we sum the perturbation series.) Thus, for any given A and B in any given field theory, only a finite set of operators contributes to the leading asymptotic behaviour; higher monomials give lower powers of q^2.

I emphasize that the fs are independent of the states $|a\rangle$ and $|b\rangle$. In particular, this means the operator product expansion is unaffected by the occurrence of spontaneous symmetry breakdown. This will be important to us later.

If A, B, and C are other than Lorentz scalars, f_{ABC} will have a nontrivial tensor structure. Since f is only a function of a single four-vector, q, it is a known tensor function of q times an unknown scalar function of q^2. It will be convenient to multiply this scalar function by a power of q^2

so that it becomes dimensionless. We will call the resulting dimensionless scalar function $\bar{f}_{ABC}$.

(3) By choosing $|a\rangle$ and $|b\rangle$ to be one-nucleon states, and A and B to be electromagnetic currents, we can use the operator product expansion to get an expression for the moments of the structure functions. The calculation is straightforward, and I do not want to do it in detail here; the result is of the form

$$F_i^n(q^2)=\sum_C d_{iC}^n \bar{f}_{ABC}(q^2)\langle a|C(0)|b\rangle_{\mathrm{R}}. \tag{6.4}$$

Here the ds are constant coefficients, terms that are less important by powers of q^2 than the terms retained have been dropped, the subscript R indicates a reduced (scalar) matrix element, and, in any given theory, for fixed i and n, the sum runs over only a finite set of Cs.

Thus the problem becomes: why do the $\bar{f}$s become constants (within experimental error) as q^2 becomes large and negative?

Because of the logarithmic polynomials mentioned before, this constant behavior is not an obvious prediction of field theory. To be specific, let us consider a theory in which there is only one coupling constant, like the standard quark–vector-gluon model. In this case, a perturbative expansion of one of the $\bar{f}$s typically yields an asymptotic expression like

$$\bar{f}=a_0+a_{11}g^2\ln q^2+a_{10}g^2+a_{22}g^4(\ln q^2)^2+\cdots \tag{6.5}$$

where the as are constant coefficients. Since we are interested in both large g (strong interactions) and large q^2 (asymptotic behaviour), this is worse than useless. Even for the (unrealistic) case of small g, Eq. (6.5) tells us nothing about asymptotic behaviour, for the largeness of the logarithm eventually overcomes the smallness of g. The only case in which we can predict asymptotic behaviour is free field theory ($g=0$); in this case, the $\bar{f}$s are indeed constants. It is for this reason that it is sometimes said that, at high negative q^2, the effects of the interactions seem to disappear, and the theory behaves as if it were free. At the moment, this may seem to you to be an excessively dramatic way of describing Bjorken scaling; nevertheless, we shall see, for a certain class of field theories, this is exactly what happens. Before we do this though, we need to develop a systematic formalism for going beyond perturbation theory and summing up the logarithms in Eq. (6.5).

6.2 *Massless field theories and the renormalization group*[28]

It can be shown, in any renormalizable field theory, to all orders of perturbation theory, that the asymptotic behaviour of the coefficient

functions in the operator product expansion is the same as it would be in a massless field theory. By a massless field theory I mean one that has only dimensionless coupling constants in its Lagrangian; not only are masses excluded but also interactions with dimensionful coupling constants, like cubic meson self-couplings. This is very plausible; the coefficient functions depend only on a single momentum, and this momentum is going to Euclidean infinity, getting as far as it can from the mass shell, and therefore losing all memory of the masses. (I emphasize that this does *not* mean that the structure functions themselves are the same as they would be in a massless theory. Eq. (6.4) contains not just $\bar{f}$, but also $\langle a|C|b\rangle$, and this stays on the mass shell.)

Thus we need only analyze the behavior of the $\bar{f}$s in a massless theory. A massless renormalizable field theory is parametrized by a set of renormalized dimensionless coupling constants, which I will call g^a. These may be either Yukawa coupling constants, quartic meson self-interaction constants, or gauge field coupling constants. In addition, another parameter is required to complete the description – a mass, M. This extra parameter is needed to *define* the others (and the scale of the renormalized fields).

Let me explain why this is so, using the simplest renormalizable field theory, $\lambda\phi^4$ theory, as an example. In the massive version of this theory, the renormalized coupling constant, λ, is usually defined as the value of $\Gamma^{(4)}$ on the mass shell, at the symmetry point, $s=t=u$. (Sometimes it is defined as the value of $\Gamma^{(4)}$ when all external momenta vanish, as in Eq. (3.17b), but this becomes the same definition when the mass vanishes.) Likewise, the renormalized field is defined as the field scaled in such a way that the derivative of $\Gamma^{(2)}$ is one on the mass shell (or, sometimes, at zero momentum). However, for a massless theory, these definitions are unworkable; all of the normal thresholds collapse on the renormalization point, and it is obviously bad policy to define λ as the value of a Green's function at the locus of an infinite number of singularities. The cure for this disease is simple; we define λ as the value of $\Gamma^{(4)}$ at some point in Euclidean space, where there are no singularities, even in the massless theory. For example, we would define λ as the value of $\Gamma^{(4)}$ at $s=t=u$, with all external momenta squared equal to $-M^2$. M can be anything; any M is as good as any other M, so long as it is not zero. Likewise, we could define the scale of the field by demanding that the derivative of $\Gamma^{(2)}$ be one when $p^2=-M^2$. These definitions can be extended in an obvious way to more complicated theories with other kinds of couplings.

Thus, the parameterization of a massless field theory requires a mass, M.

But M is arbitrary; in a given physical theory, if you change the value of M, this can always be compensated for by an appropriate change in the gs and in the scales of the renormalized fields, because the only function of M is to define these quantities. Phrased in equations, if we make a small change in M,

$$M \to M(1+\varepsilon), \tag{6.6a}$$

where ε is infinitesimal, this can always be compensated for by an appropriate small change in the coupling constants

$$g^a \to g^a + \beta^a \varepsilon, \tag{6.6b}$$

and a corresponding small change in the scale of renormalized operators, e.g.

$$A(x) \to (1+\gamma_A \varepsilon) A(x). \tag{6.6c}$$

By dimensional analysis, the βs and γs can depend only on the gs,

$$\beta^a = \beta^a(g),\ \gamma_A = \gamma_A(g), \tag{6.7}$$

where, to simplify notation, a single g in the argument of a function stands for all the gs. If A is one of a set of operators that can mix with one another as a result of renormalization (as is the case, for example, with ϕ^4 and $\partial_\mu \phi \partial^\mu \phi$), Eq. (6.6c) should be replaced by a matrix equation. However, for simplicity, we will ignore this possible complication here.

The infinitesimal transformations (6.6) define a one-parameter group, called the renormalization group. All physical quantities must be invariant under this group. In particular, the $\bar{f}$s must be invariant; thus

$$\left[M \frac{\mathrm{d}}{\mathrm{d}M} + \beta^a(g) \frac{\partial}{\partial g^a} + \gamma_{ABC}(g) \right] \bar{f}_{ABC} = 0. \tag{6.8}$$

where

$$\gamma_{ABC} = \gamma_A + \gamma_B - \gamma_C. \tag{6.9}$$

Of course, similar equations can be derived for any other object in the theory, in particular, for Green's functions. Only the γ-terms depend on the object under consideration.

Since these renormalization-group equations are exactly valid, they must be valid order-by-order in renormalized perturbation theory. Thus, from perturbation expansions of Green's functions, it is possible to deduce perturbation expansions for the βs and γs. If this is done for the quark–vector-gluon model, for example, one finds that the power series for β begins with terms of order g^3, while those for either the quark or gluon γ begin with terms of order g^2. This is reasonable, because β reflects the

effects of coupling-constant renormalization, which begin at order g^3, while γ reflects those of wave-function renormalization, which begin at order g^2.

6.3 *Exact and approximate solutions of the renormalization group equations*

The differential equations of the renormalization group are a mathematical expression of a physical triviality, that the only function of the mass M is to define the renormalized coupling constants and the scale of the renormalized fields. Nevertheless, they can, in favorable circumstances, be used to obtain highly non-trivial information about the asymptotic behaviour of the theory. The basic reason for this is simple dimensional analysis; since $\bar{f}$ is dimensionless,

$$\bar{f}_{ABC} = \bar{f}_{ABC}(Q/M, g), \tag{6.10}$$

where $Q=(-q^2)^{\frac{1}{2}}$. Thus, knowledge of the (trivial) dependence on M is equivalent to knowledge of the (non-trivial) dependence on Q.

To work this out in detail, let me assume that we know the βs and γs exactly. Then there is a standard method[29] for solving the linear partial differential equation

$$\left[M\frac{\partial}{\partial M}+\beta^a(g)\frac{\partial}{\partial g^a}+\gamma(g)\right]\bar{f}(Q/M, g)=0, \tag{6.11}$$

where I have suppressed the ABC subscript for notational simplicity. The standard method goes in two steps. First, one constructs $g'^a(g, t)$, a set of functions of the gs and a single extra variable, t, defined as the solution to the ordinary differential equations

$$\mathrm{d}g'^a/\mathrm{d}t=\beta^a(g'), \tag{6.12a}$$

with the boundary condition

$$g'^a(g, 0)=g^a. \tag{6.12b}$$

Then, the general solution to Eq. (6.11) is

$$\bar{f}=F(g'(g, \ln[Q/M]))\times\exp\int_0^{\ln[Q/M]}\gamma(g'(g, t))\mathrm{d}t \tag{6.13}$$

where F is an arbitrary function. Thus we see the power of the renormalization group; if we know everything for all gs at $Q=M$, then we know everything for all gs at all Qs.

Unfortunately, we do not know everything for all gs. Typically, we only know the first few terms in a power series in g. Even in this case, though, it is possible to use the renormalization group to squeeze out extra information. To show how this is done, let me return to the quark

Abelian-vector-gluon model. Here I have argued that

$$\beta(g) = bg^3 + O(g^5). \tag{6.14}$$

where b is a numerical coefficient. The only thing I will ask you to take on trust is that, if you actually do the relevant Feynman calculations, you will find that b is positive. Eq. (6.12) then becomes

$$dg'/dt = bg'^3 + O(g'^5). \tag{6.15}$$

Now let us attempt to construct an approximate solution of this equation, for small g, by ignoring the terms of order g^5. The solution is trivially obtained by quadratures,

$$\frac{1}{g'^2} = \frac{1}{g^2} - 2bt, \tag{6.16}$$

or

$$g'^2 = \frac{g^2}{1 - 2btg^2}. \tag{6.17}$$

When can we trust this approximate solution? When t gets large and positive, the approximate g' becomes large, and the terms we have neglected become comparable to the terms we have retained. For this range of t, the approximation is garbage. On the other hand, as t becomes large and negative, the approximate g' becomes smaller and smaller, and the terms we have neglected therefore become smaller and smaller than the terms we have retained. For this range of t, the approximation is wonderful.

Now, when we plug g' into Eq. (6.13), t becomes $\ln(Q/M)$. Thus our approximation gets better and better the smaller Q is. Furthermore, we can improve on it as much as we want, simply by doing more perturbation calculations to get the higher terms in the expansions of β, γ, and F.

To phrase the whole thing more generally, an ordinary perturbation expansion, like (6.5), has two conditions for its reliability, $|g| \ll 1$ and $|\ln(Q/M)| \ll 1$. The approximation scheme I have described replaces these with a single condition, $|g'| \ll 1$. This single condition may hold in regions where the logarithm is large; in the case at hand, this includes the region of arbitrarily large negative $\ln(Q/M)$.

This is marvelous stuff; the renormalization group has tamed the logarithms in Eq. (6.5). Unfortunately, this is of no physical interest, for two reasons. (1) To start the approximation, g must be small. We are interested in strong interactions. (2) We can tame the logarithms in the region of small Q, the infrared region. We are interested in large Q, the ultraviolet region. Indeed, our whole method of approach is nonsense in the region of small Q, because, when Q is small, it is no longer sensible to neglect particle masses.

Now let us do another example, pure Yang–Mills theory for some simple Lie group. Here again there is only one coupling constant, and coupling-constant renormalization begins in order g^3, so this is hardly a new example. Everything will be exactly the same as for the quark–vector-gluon model; the only possible difference can be in the value of b, the constant in Eq. (6.15). I now announce the great discovery of the last year: b is *negative*.[30] (This is true whatever the simple Lie group.)

Thus, our previous analysis is turned on its head. Large negative t is replaced by large positive t, infrared by ultraviolet. There exists a family of renormalizable field theories for which the logarithms can be tamed in the ultraviolet region! In this region, we obtain, from lowest-order perturbation theory and the renormalization group, an approximation that gets better and better as Q gets larger and larger. Furthermore, we can improve on the approximation as much as we want, simply by doing more perturbation calculations to get the higher terms in the expansions of β, γ, and F.

6.4 *Asymptotic freedom*

What we have discovered for pure Yang–Mills theory is a special case of a phenomenon called asymptotic freedom. A general renormalizable field theory is said to be asymptotically free if, for small g^a,

$$\lim_{t\to\infty} g^{a\prime}(g, t)=0. \tag{6.18}$$

All my remarks for pure Yang–Mills theory carry over without alteration to a general asymptotically free theory; in particular, asymptotic behaviour for large Q is exactly computable from simple perturbation theory and the renormalization group. In principle, it is simple to test whether any given field theory is asymptotically free; all one needs to do is compute the β-functions to lowest non-vanishing order, and then solve the differential equations (6.12). In practice, the test is difficult to carry out; the computation of the β-functions is straightforward, but, in the typical case, the differential equations can not be solved analytically, and one has to resort to tedious case-by-case numerical integration. Thus, although many asymptotically free theories have been discovered, and a few general theorems have been proved, we have nothing like a complete classification of asymptotically free theories. I will tell you what is known about the classification problem shortly; first, though, I would like to convince you that asymptotic freedom offers a possible explanation of Bjorken scaling.

At first thought, this is a preposterous suggestion. Asymptotic freedom is a property of field theories for small coupling constants, and Bjorken

scaling is a strong-interaction effect. Nevertheless, it is possible, with a little hand-waving, to establish a connection. For simplicity, let us consider an asymptotically free theory with only one coupling constant, g. By assumption, β is negative for small positive g. Let us denote the first positive zero of β by g_1. We certainly can not compute g_1 perturbatively; if we were asked to guess, we would probably guess that g_1 was something like 1 or π or maybe even infinity (if β has no zeros). In any case, it certainly can not be a small number; for small coupling constants, we trust perturbation theory, and perturbation theory tells us β is negative. Whatever the value of g_1, for any g less than g_1, β is negative. Therefore, if we start from such a g, and integrate

$$dg'/dt = \beta(g'), \tag{6.19}$$

g' will decrease. As we continue to integrate the equation, it will continue to decrease, until we finally reach the region of small g', where formulae like Eq. (6.17) will be valid. Thus, the asymptotic expressions derived from renormalization-group-improved perturbation theory are valid for theories defined by large coupling constants as well as small. If we are very lucky, and β has no positive zeros, they will be valid for all values of g.

The decrease from large to small g' can be quite rapid. As an example, let us take the result of the lowest-order perturbation theory, Eq. (6.16), and imagine that it is valid for large coupling constants as well as small. (I emphasize that this is undoubtedly false; I am just using it as a simple model of rapid decrease.) For a pure Yang–Mills theory with gauge group SU(3),

$$b = -11/16\pi^2. \tag{6.20}$$

Thus, Eq. (6.16) becomes

$$\left(\frac{g'^2}{4\pi}\right)^{-1} = \left(\frac{g^2}{4\pi}\right)^{-1} + \frac{11t}{2\pi}. \tag{6.21}$$

Now let us imagine that we start out with some very large value of $g^2/4\pi$, say 10^3, at $t=0$. Then by going to $t=1$ (that is to say, by increasing Q by a factor of e) we arrive at $g'^2/4\pi = 2\pi/11$. From this point on, the variation is quite slow; multiplying Q by e again merely halves g', and multiplication by e^2 is required to halve it again. Thus we are led to conjecture a qualitative picture in which a very large value of g' at low momentum zooms down with lightning rapidity to a small value, and then inches its way to zero.

What sort of asymptotic behaviour do we predict, once we are in the region of small g'? To evaluate Eq. (6.13), we need to know not only g',

but also γ. For small g',

$$\gamma(g') = cg'^2 + o(g'^4), \tag{6.22}$$

where c is a numerical coefficient. From Eq. (6.17), for large t,

$$g'^2 \approx -1/2bt, \tag{6.23}$$

whence,

$$\gamma \approx -c/2bt. \tag{6.24}$$

Thus, the significant variation in Eq. (6.13) comes from the upper limit in the integral; for large t,

$$\bar{f} \approx K[\ln(Q/M)]^{-c/2b} \tag{6.25}$$

where K is a constant. This is *not* Bjorken scaling; the moments of the structure functions are not constants, but powers of logarithms. Nevertheless, a power of a logarithm is a very slowly varying function. I have not studied the SLAC–MIT data myself, but I am told by those who have looked at them (with optimistic eyes) that they can be fit as well with powers of logarithms as with constants.

Note that, for any given model, these powers can be computed by lowest-order perturbative calculations.[31] For example, the popular colored-quark model, with a color octet of vector gluons, is an asymptotically free theory with only one coupling constant. In this model, the moments of the isospin-odd (proton minus neutron) transverse structure function have the asymptotic form

$$F^{(n)} \propto (\ln Q)^{[-0.296\ln(n+2)+0.051]}. \tag{6.26}$$

These are rather small powers for small moments (-0.2 for $n=0$), and grow slowly with n, reaching -1 only for $n=27$. Of course, since we do not know the constant coefficients of the moments, we can not reconstruct the structure functions from formulae like Eq. (6.26). However, it is easy to construct functions whose moments obey Eq. (6.26) and which display quite small deviations from scaling except for x very near to 1. (The very high moments are obviously sensitive only to the behavior of $F(q^2, x)$ in this neighborhood.)

If we accept asymptotic freedom as the explanation of Bjorken scaling, then, whatever the field theory of the strong interactions, it must be asymptotically free. (Bjorken scaling places no restrictions on the weak and electromagnetic interactions; these are negligible in the relevant energy region.) Thus, it is important to know what field theories are asymptotically free. Here is what we know now:

(1) All pure Yang–Mills theories based on groups without Abelian factors are asymptotically free.[30]

(2) Theories of non-Abelian gauge fields and Fermi multiplets are sometimes asymptotically free and sometimes not. The fermions make a positive contribution to the β-function; if the theory has too many fermions, the sign of β is reversed and asymptotic freedom is lost. 'Too many' is typically a large number. For example, if the gauge group is SU(3), sixteen triplets of fermions are not too many.[30]

(3) Much less is known about theories of non-Abelian gauge fields and scalar multiplets; typically, these theories involve a large number of quartic meson coupling constants, and this makes the investigation of the differential equations difficult. There are some theories involving scalar fields which are known to be asymptotically free.[32] At the moment, there are no known asymptotically free theories for which all the gauge mesons may be given a mass by scalar vacuum expectation values. I do not view this as a serious difficulty, for two reasons. (1) The investigation is still in its early stages; such a theory may be found next week. (2) Even if no such theory is found, we are talking about models of the strong interactions; although the couplings may become weak at large momentum, they are certainly strong at small momentum, and this is where spontaneous symmetry breakdown occurs. Therefore, symmetry breakdown might well occur non-perturbatively, as discussed in Section 2.6.

(4) Any renormalizable field theory that does not involve non-Abelian gauge fields is not asymptotically free.[33]

This last result has far-reaching consequences: if we accept asymptotic freedom as the explanation for Bjorken scaling, then the field theory of the strong interactions must be asymptotically free. If it is to be asymptotically free, then it must involve non-Abelian gauge fields. Since no-one has ever seen a massless hadron, these gauge fields must acquire masses. The only known mechanism by which gauge fields can acquire masses is through spontaneous symmetry breakdown. Thus, *the field theory of the strong interactions must be a spontaneously broken gauge field theory.*

This is a striking conclusion, suggestive of deep connections between the strong and weak interactions. It implies a complete reversal of the conventional wisdom of only a few years ago. We used to believe that at high (Euclidean) energies the weak interactions became strong; now we believe that the strong interactions become weak.

6.5 *No conclusions*

I know of no way to put a proper conclusion to these lectures, because I know of no way to judge the validity of the ideas we have discussed. They are certainly ideas of great beauty, and they certainly resolve many long-standing theoretical problems, but they equally

certainly have not yet quantitatively confronted experiment. Spontaneously broken gauge field theories are in the uncomfortable position of SU(3) without the Gell-Mann–Okubo formula, or current algebra without the Adler–Weisberger relation. There are good reasons for this, which I explained in the Introduction, but still one can not help feeling nervous. It is very possible that this whole beautiful and complex structure will be swept into the dustbin of history by a thunderbolt from Batavia. All we can do is wait and see.

Appendix: One-loop effective potential in the general case

This appendix is a computation of the one-loop effective potential, V, for a general renormalizable field theory. Such a theory contains three types of interactions: spinless-meson self-interactions, Yukawa couplings of mesons and fermions, and gauge-field interactions. We shall see that in an appropriate gauge (Landau gauge), these three types of interactions contribute additively to the effective potential in one-loop approximation; thus,

$$V = U + V_{\mathrm{m}} + V_{\mathrm{f}} + V_{\mathrm{g}} + V_{\mathrm{ct}}, \tag{A.1}$$

where the first term is the zero loop effective potential, the next three terms are the contributions from the three types of interactions, and the last term is a quartic polynomial in ϕ_{c}, the finite residue of the renormalization counterterms, determined once we state our renormalization conditions. The method of computation will be a direct generalization of the diagrammatic summation of Section 3.[34]

(1) *Spinless-meson contribution.* Here the analysis is almost identical to that of Section 3; the only difference is that there may be many meson fields. Thus each internal line in the graphs of Fig. 5 carries an index a, b, etc., labeling the meson field, and the black dots represent matrices for the transition from a meson of type a to one of type b:

$$\mathrm{i}[U''(\phi_{\mathrm{c}})]_{ab} = \mathrm{i}\partial^2 U/\partial\phi^a\partial\phi^b|_{\phi=\phi_{\mathrm{c}}}. \tag{A.2}$$

In computing the graphs, we must not only integrate over the internal momentum, but also sum over the internal indices. This is equivalent to multiplying the matrices around the loop and then taking the trace. Thus, from Eq. (3.33), we obtain

$$V_{\mathrm{m}} = \frac{1}{64\pi^2}\,\mathrm{Tr}([U''(\phi_{\mathrm{c}})]^2 \ln U''(\phi_{\mathrm{c}})). \tag{A.3}$$

(2) *Fermion contribution.* Here again the graphs are almost the same as in Fig. 5; the only difference is that the internal lines are fermion lines.

(Thus, you should imagine them as carrying arrows.) The relevant term in the Lagrangian is

$$\mathscr{L} = i\bar{\psi}^a \partial \psi^a + \bar{\psi}^a m_{ab}(\phi)\psi^b + \cdots. \tag{A.4}$$

Here m is the sum of two terms: a constant term (the fermion masses) and a term linear in ϕ (the Yukawa couplings). It can also be broken into two parts in a different way:

$$m = A + iB\gamma_5. \tag{A.5}$$

(I use a Hermitian γ_5.) The reality of the Lagrangian implies that A and B are Hermitian matrices. I have chosen the name m for this matrix because $m(\langle\phi\rangle)$ is the fermion mass matrix, in zero-loop approximation.

The computation can be made to look like the preceding one by grouping the terms in pairs:

$$\cdots m \frac{1}{\not p} m \frac{1}{\not p} \cdots = \cdots m m^\dagger \frac{1}{p^2} \cdots. \tag{A.6}$$

Now the only differences between the fermion computation and the boson one are: (1) The combinatoric factor of $\frac{1}{2}$ is missing because the lines have arrows on them, and thus the graphs are not invariant under reflections. (2) This is compensated for by the fact that the odd terms in the infinite series vanish when we take the trace on Dirac indices. (3) There is an overall Fermi minus sign. Thus we obtain

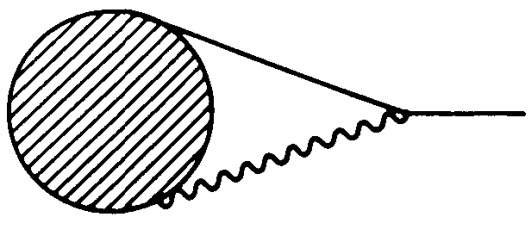
$$V_f = -\frac{1}{64\pi^2} \operatorname{Tr}([m m^\dagger(\phi_c)]^2 \ln m m^\dagger(\phi_c)). \tag{A.7}$$

Note that here the trace is on Dirac indices as well as internal indices.

(3) *Gauge-field contribution.* If we work in a general gauge, the trilinear coupling between gauge fields and spinless mesons can lead to troublesome graphs of the form shown in Fig. 6. (Here the straight line is a spinless meson, and the wiggly line a gauge field.) However, if we work in Landau gauge,

$$D_{\mu\nu} = i\frac{g_{\mu\nu} - k_\mu k_\nu / k^2}{k^2} \tag{A.8}$$

Fig. 6



these graphs vanish. This is because the external meson carries zero momentum; the sum of the meson momenta is the same as the gauge-field momentum, and gives zero when we contract it with the propagator (A.8). Hence we need only worry about the quadrilinear coupling

$$\mathscr{L} = \cdots + \tfrac{1}{2} A_{\mu a} A_b^{\mu} M_{ab}^2(\phi) + \cdots, \tag{A.9}$$

where

$$M_{ab}^2 = g_a g_b (T_a \phi) \cdot (T_b \phi), \tag{A.10}$$

and g_a is the coupling constant of the ath gauge field. $M^2(\langle\phi\rangle)$ is the gauge-meson mass-squared matrix in zero-loop approximation, whence its name.

The computation is now identical with the preceding case, except that it is now gauge fields that run around the loop. Thus,

$$V_g = \frac{3}{64\pi^2} \operatorname{Tr}([M^2(\phi_c)]^2 \ln M^2(\phi_c)). \tag{A.11}$$

The factor of three comes from the trace of the propagator (A.8).

Notes and references

1. There are many excellent reviews that can be used to rectify these lapses: A. De Rujula, in *Proceedings of the First International Meeting on Fundamental Physics*; E. S. Abers and **B.** W. Lee, *Physics Reports* **9**, 1 (1973); J. R. Primack and H. R. Quinn (in the Proceedings of the 1973 Santa Cruz Summer School on Particle Physics); C. H. Llewellyn Smith, in *Proceedings of the 1973 Scottish Universities Summer School*, ed. R. Crawford and R. Jennings (Academic Press, 1974); M. Veltman in *Proceedings of the 1973 Bonn Conference*, (North Holland, 1974). This last contains a meticulous history.
2. This idea goes back to the classic work of Goldstone, Nambu, and Jona-Lasinio. J. Goldstone, *Nuovo Cimento* **19**, 15 (1961); Y. Nambu and G. Jona-Lasinio, *Phys. Rev.* **122**, 345 (1961); **124**, 246 (1961).
3. Notation: the signature of the metric tensor is $(+ - - -)$; $\partial_\mu = \partial/\partial x^\mu$; summation over repeated indices is always implied.
4. J. Goldstone, A. Salam, and S. Weinberg, *Phys. Rev.* **127**, 965 (1962).
5. D. Kastler, D. Robinson, and J. Swieca, *Comm. Math. Phys.* **3**, 151 (1966).
6. F. Englert and R. **Brout**, *Phys. Rev. Letters* **13**, 321 (1964); P. Higgs, *Phys. Letters* **12**, 132 (1964); G. Guralnik, C. Hagen and T. Kibble, *Phys. Rev. Letters* **13**, 585 (1964); P. Higgs, *Phys. Rev.* **145**, 1156 (1966); T. Kibble, *Phys. Rev.* **155**, 1554 (1967).
7. C. N. Yang and R. Mills, *Phys. Rev.* **96**, 191 (1954); R. Utiyama, *Phys. Rev.* **101**, 1597 (1956); S. Glashow and M. Gell-Mann, *Ann. Phys.* (N.Y.) **15**, 437 (1961).
8. For some tantalizing recent explorations, see R. Jackiw and K. Johnson; *Phys. Rev.* **D8**, 2386 (1973), and J. Cornwall and R. Norton, *Phys. Rev.* **D8**, 3338 (1973); J. Cornwall, *Phys. Rev.* **D10**, 500 (1974). See also R. Jackiw, in *Laws of Hadronic Structure*, 1973 Ericé Summer School, Academic Press (1975).
9. J. Schwinger, *Phys. Rev.* **128**, 2425 (1962); W. Thirring and J. Wess, *Ann. Phys.* (N.Y.) **27**, 331 (1964); J. Lowenstein and J. Swieca, *Ann. Phys.* (N.Y.) **68**, 172

(1971). This model has another amusing feature; it possesses a continuous (non-gauge) symmetry (chirality) that is not associated with a local conserved current, because of an anomaly. This symmetry is spontaneously broken but there is no Goldstone boson, because there is no conserved local current.

10. For more details (and references) see Chapter 4 in this volume.
11. Ward identities: A. Slavnov, *Theo. and Math. Phys.* **10**, 99 (1972). Gauge-invariant cutoff: G. 't Hooft and M. Veltman, *Nucl. Phys.* **44B**, 189 (1973). Renormalization: G. 't Hooft, *Nucl. Phys.* **B33**, 173 (1971); **B35**, 167 (1971); **B**. Lee and J. Zinn-Justin, *Phys. Rev.* **D5**, 3121, 3137, 3155 (1972). A detailed review is Abers and Lee.[1]
12. This, and much of what follows in this section, is plagiarized from S. Coleman and E. Weinberg, *Phys. Rev.* **D7**, 1888 (1973), which contains more details, references, and applications. The effective potential was introduced by Goldstone, Salam, and Weinberg,[4] and by G. Jona-Lasinio, *Nuovo Cimento* **34**, 1790 (1964).
13. This can most easily be proved with the aid of functional integration; see the discussion sections.
14. There is a parameter, which we have set equal to one by our choice of units, that enters the theory in the same way as a; this is h. Thus, it is sometimes said that the loop expansion is an expansion in powers of h. Y. Nambu, *Phys. Letters* **26B**, 626 (1966).
15. The computation of the one-loop effective potential can be done along the same lines for a general renormalizable field theory, involving arbitrary numbers of spinless mesons, fermions, and gauge fields, with hardly more labor. For the interested reader, the computation is done in the Appendix. The computation may also be done with functional integrals; this method avoids the infinite summation of diagrams. See Lee and Zinn-Justin[11] and R. Jackiw, *Phys. Rev.* **D9**, 1686 (1974).
16. This is a result of K. Symanzik, *Comm. Math. Phys.* **16**, 48 (1970).
17. This fits in with the loop expansion being an expansion in powers of h.[14] You should not allow this discussion to obscure one big difference between particle mechanics and field theory: in particle mechanics, there is no spontaneous symmetry breakdown, even for a double-welled potential of the sort shown in Fig. 2. The difference can easily be seen in the simple quadratic approximation discussed above. In particle mechanics, if we approximate the two supposed degenerate ground states by harmonic-oscillator wave functions, they have a non-zero inner product. This induces mixing which breaks the degeneracy; only one linear combination of the two states is a true ground state. It is easy to make the corresponding approximation in field theory, if we put the theory in a box of volume V. A simple computation then shows that the inner product goes to zero exponentially as the volume goes to infinity, and (at least in this approximation) the degeneracy remains.
18. This phenomenon is well-known; it occurs in the classic calculation of the effective action for electrons coupled to a constant external electromagnetic field. (W. Heisenberg and H. Euler, *Z. Physik* **98**, 714 (1936); J. Schwinger, *Phys. Rev.* **82**, 664 (1951).) This is real for a magnetic field, but imaginary for an electric field; the reason is that, in the presence of a constant electric field, the vacuum is unstable and decays into electron–positron pairs.
19. S. Weinberg, *Phys. Rev. Letters* **29**, 1698 (1972).
20. The application of functional integrals to quantum mechanics is due to Feynman, and the standard physics text is R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, 1965). This follows Feynman's original definition of the functional integral. This is apparently different from

the definition I will give, but the two can be shown to be equivalent. A good mathematical reference is I. Gelfand and N. Vilenkin, *Generalized Functions*, Vol. 4 (Academic Press, 1964).

21. The clearest derivation I know of is that of K. S. Cheng, *J. Math. Phys.* **13**, 1723 (1972). Cheng evaluates the functional integral *à la* Feynman,[20] and shows that it defines the same dynamics as the Schrödinger equation with an appropriate ordering of the *p*s and *q*s.
22. The Hamiltonian form first appears in Appendix **B** of R. P. Feynman, *Phys. Rev.* **84**, 108 (1951). It was subsequently rediscovered many times by many authors, but I can not find any reference where its dangerous ambiguities are discussed. Indeed, one commonly finds in the literature the false statement that the Hamiltonian form is a method of quantization that is invariant under general classical canonical transformations. (A possible exception is the work of **B**. S. DeWitt, *Rev. Mod. Phys.* **29**, 377 (1957). Those portions of this paper that I can understand I believe to be correct.) The remarks in the text are the product of conversations with D. Gross, C. Callan, and S. Treiman.
23. L. Faddeev and V. Popov, *Phys. Letters* **25B**, 29 (1969) and 'Perturbation theory for gauge invariant fields', Kiev lecture available in English as NAL-Thy-57.
24. R. Arnowitt and S. Fickler, *Phys. Rev.* **127**, 1821 (1962).
25. A good review of the anomalies is R. Jackiw, in *Lectures on Current Algebra and its Applications* (Princeton U. Press, 1970). Implications for gauge theories are discussed in D. Gross and R. Jackiw, *Phys. Rev.* **D6**, 477 (1972) and W. A. Bardeen, in *Proceedings of the XVI International Conference on High Energy Physics*, Vol. 2 (NAL, 1972).
26. K. Fujikawa, **B**. Lee, and A. Sanda, *Phys. Rev.* **D6**, 2923 (1972).
27. So many people have made important contributions to this standard lore that a fair set of references would be longer than Sec. 6.1. A good brief review is C. G. Callan in *Proceedings of the International School of Physics "E. Fermi" Course LIV* (Academic Press, 1972).
28. More details, and references, can be found in Chapter 3 in this volume.
29. For a detailed derivation of the solution, see Note 28.
30. D. Gross and F. Wilczek, *Phys. Rev. Letters* **30**, 1343 (1973); H. D. Politzer, *ibid.*, 1346. For pure Yang–Mills theory, the result was known to G. 't Hooft in the summer of 1972, but not published by him.
31. For detailed computations see D. Gross and F. Wilczek, *Phys. Rev.* **D8**, 3633 (1973); **D9**, 980 (1974); and H. Georgi and H. D. Politzer, *Phys. Rev.* **D9**, 416 (1974).
32. Gross and Wilczek.[30,31]
33. S. Coleman and D. Gross, *Phys. Rev. Letters* **31**, 851 (1973).
34. The computation can also be done by functional integration.[15]